The equation of state of geometric dark energy in $f(R)$ cosmology

(for more details see arXiv: 1312.5428, and 1206.1642)

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3. $f(R)$ metric gravity
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Goal

The problem or issue that I'll be tackling in this talk is not new, but perhaps has not been stressed enough. It's about the (possible) ambiguity that may arise when insisting in trying to define the EOS associated with the "(geometric) dark energy" that results when one puts forward alternative theories of gravity, namely, $f(R)$ theories or STT to explain the accelerated expansion of the Universe as opposed to the cosmological constant.

Introduction

– ΛDM paradigm within GR: the simplest and perhaps most successful cosmological model.

– Alternative Theories of Gravity: try to "replace" Dark components. This is one among several possibilities (e.g. inhomogeneous models within GR). More complicated, but it's a worth exploring possibility (I skip the heuristic and philosophyical arguments on the "problems" of Λ. Perhaps you will hear more on this during this meeting.)

– $f(R)$ metric theories of gravity: a possible explanation for the accelerated expansion of the Universe as opposed to the Cosmological Constant. (As far as we know, DM **must** be considered, otherwise it seems impossible to recover the rest of cosmological observations.). These alternative theories of gravity (like others) allows for an "EOS of (geometric) dark energy" that varies in cosmic time, unlike Λ. Finally, I won't have time to discuss other gravitational tests that $f(R)$ theories should pass. If some of you are interested, we can talk about these during the meeting.

$f(R)$ METRIC GRAVITY

$$S[g_{ab},\psi] = \int \frac{f(R)}{2\kappa}\sqrt{-g}\, d^4x + S_{\rm matt}[g_{ab},\psi]\ , \tag{1}$$

where R =Ricci scalar, $f(R)$ an *a priori* arbitrary function of R, $\kappa := 8\pi G_0$, and ψ represents the matter fields – ordinary and DM – (here $c = 1$).
Varying the action Eq. (1) with respect to the metric yields

$$f_R R_{ab} - \frac{1}{2} f g_{ab} - (\nabla_a \nabla_b - g_{ab}\Box)\, f_R = \kappa T_{ab}\ , \tag{2}$$

where $f_R := \partial_R f$, $\Box = g^{ab}\nabla_a\nabla_b$, and T_{ab} is the EMT of matter.

When expanding the derivative ∇ acting on f_R and taking the trace we obtain:

$$\Box R = \frac{1}{3f_{RR}}\left[\kappa T - 3f_{RRR}(\nabla R)^2 + 2f - Rf_R\right] , \tag{3}$$

where $(\nabla R)^2 := g^{ab}(\nabla_a R)(\nabla_b R)$, and $T := T^a{}_a$. Using this equation in the field equation to replace $\Box R$ we find

$$G_{ab} = \frac{1}{f_R}\left[f_{RR}\nabla_a\nabla_b R + f_{RRR}(\nabla_a R)(\nabla_b R) - \frac{g_{ab}}{6}\left(Rf_R + f + 2\kappa T\right) + \kappa T_{ab}\right] . \tag{4}$$

We shall be dealing with equations (3) and (4).

$f(R)$ theories can mimic Λ:

Notice that in vacuum, $R = R_1 = const.$ is a solution of

$$\Box R = \frac{1}{3f_{RR}}\left[\kappa T^{=0} - 3f_{RRR}(\nabla R)^2 + 2f - Rf_R\right] , \tag{5}$$

provided R is a root of $V'(R) = 2f - Rf_R$ (i.e. $2f(R_1) = R_1 f_R(R_1)$), assuming $f_{RR}(R_1) \neq 0$. That is, R_1 is a critical point (e.g. maximum or minimum) of the "potential" $V(R)$. In such an instance, the field equation

$$G_{ab} = \frac{1}{f_R}\left[f_{RR}\nabla_a\nabla_b R + f_{RRR}(\nabla_a R)(\nabla_b R) - \frac{g_{ab}}{6}\left(Rf_R + f + 2\kappa T^{=0}\right) + \kappa T^{=0}_{ab}\right] . \tag{6}$$

reduces to

$$G_{ab} = -g_{ab}\frac{R_1}{4} \qquad \text{(in vacuum)} . \tag{7}$$

Therefore $f(R)$ theory behaves like GR with an *effective cosmological constant* $\Lambda_{eff} = R_1/4$!

The fact that the theory can admit this solution for R allows one to find non-trivial solutions that asymptotically (past,future, or spatial infinity) match a De Sitter solution, which in turn can explain several cosmological observations.

Cosmology

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right] \quad , \tag{8}$$

where $k = \pm 1, 0$. Later we focus only on $k = 0$.

$$H^2 + \frac{k}{a^2} + \frac{1}{f_R}\left[f_{RR}H\dot{R} - \frac{1}{6}\left(f_R R - f\right)\right] = \frac{\kappa\rho}{3f_R} \quad , \tag{9}$$

$$\dot{H} = -H^2 + \frac{1}{f_R}\left(f_{RR}H\dot{R} + \frac{f}{6} - \frac{\kappa\rho}{3}\right) \quad , \tag{10}$$

where $H = \dot{a}/a$ is the Hubble expansion. From Eq. (3) we find

$$\ddot{R} = -3H\dot{R} - \frac{1}{3f_{RR}}\left[3f_{RRR}(\dot{R})^2 + \underbrace{2f - f_R R}_{V'(R)} - \kappa(\rho - 3p)\right] \quad . \tag{11}$$

If during the cosmic evolution R reaches an extrema of the potential $V(R)$ (with vanishing $\dot{R}$), say, at present time where the matter contributions ρ, p are "small" compared to $\rho_{\rm crit}$ then $R \approx R_1$ with $V'(R_1) = 0$, and so $\dot{H} + H^2 = \ddot{a}/a \sim R_1/12 = \Lambda_{\rm eff}/3 > 0$ if $\Lambda_{\rm eff} > 0$. Thus $\ddot{a} > 0 \rightarrow$ Accelerated expansion !!. This what happens precisely when solving the full equations numerically taking into account all the matter terms.

The EOS of geometric dark energy

For cosmological applications it is sometimes useful to write the $f(R)$ field equations as "Einstein" field equations with a total-effective EMT: $G_{ab} = \kappa T^{\rm tot}_{ab}$

$$\kappa T^{\rm tot}_{ab} = \frac{1}{f_R}\left[f_{RR}\nabla_a\nabla_b R + f_{RRR}(\nabla_a R)(\nabla_b R) - \frac{g_{ab}}{6}\left(Rf_R + f + 2\kappa T\right) + \kappa T_{ab}\right] .$$

Because the EMT of matter T_{ab} itself is mixed in a non-trivial way with $f(R)$ factors, thus there is non canonical way of defining the EMT of "geometric dark energy":

$$\tilde{T}^X_{ab}(A,B) := AT^{\rm tot}_{ab} - BT_{ab} \ , \tag{12}$$

Depending on the values adopted for the scalars A and B (see next slide) the defintion of the GDE EMT changes from author to author.
Alternatively, one can write $\tilde{T}^X_{ab}(A,B)$ in terms of purely geometrical quantities:

$$\tilde{T}^X_{ab}(A,B) = \kappa^{-1}\left(AG_{ab} - B\mathfrak{G}_{ab}\right) \ . \tag{13}$$

where

$$\mathfrak{G}_{ab} = f_R G_{ab} - f_{RR}\nabla_a\nabla_b R - f_{RRR}(\nabla_a R)(\nabla_b R) + g_{ab}\left[\frac{1}{2}\left(Rf_R - f\right) + f_{RR}\Box R + f_{RRR}(\nabla R)^2\right]$$

EMT of GDE	energy-density, pressure and EOS of GDE
$\tilde{T}^{X}_{ab}(A,B) := A T^{\rm tot}_{ab} - B T_{ab}$	$\tilde{\rho}_X = \frac{A}{\kappa f_R}\left[\frac{1}{2}\left(f_R R - f\right) - 3 f_{RR} H \dot{R} + \kappa\rho\left(1 - \frac{B f_R}{A}\right)\right]$ $\tilde{p}_X = -\frac{A}{3\kappa f_R}\left[\frac{1}{2}\left(f_R R + f\right) + 3 f_{RR} H \dot{R} - \kappa\left(\rho - 3 p_{\rm rad}\frac{B f_R}{A}\right)\right]$ $\tilde{\omega}_X = \frac{\tilde{p}_X}{\tilde{\rho}_X}$

Definition	A	B	EMT
I $(T^{X}_{ab},\ \rho_X,\ p_X,\ \omega_X)$	1	1	T^{X}_{ab} conserved
II $(T^{II\,,\,X}_{ab},\ \rho^{II}_X,\ p^{II}_X,\ \omega^{II}_X)$	f^0_R	1	$T^{II\,,\,X}_{ab}$ conserved
III $(T^{III\,,\,X}_{ab},\ \rho^{III}_X,\ p^{III}_X,\ \omega^{III}_X)$	f_R	1	$T^{III\,,\,X}_{ab}$ not conserved
IV $(T^{IV\,,\,X}_{ab},\ \rho^{IV}_X,\ p^{IV}_X,\ \omega^{IV}_X)$	1	f_R^{-1}	$T^{IV\,,\,X}_{ab}$ not conserved (conserved only in vacuum)

Now, the X–EMT of the Definitions I and II are conserved because $T_{ab}^{\rm tot}$ is conserved (due to the Bianchi identities) and one can prove that the EMT T_{ab} of matter alone is also conserved. In particular,

$$\dot{\rho}_X^i + 3H\left(\rho_X^i + p_X^i\right) = 0 \ . \tag{14}$$

(for $i = I, II$).
A corollary is that the EMT of Definitions III and IV are not conserved: therefore

$$\dot{\rho}_X^i + 3H\left(\rho_X^i + p_X^i\right) \neq 0 \ . \tag{15}$$

(for $i = III, IV$).
This is rather unpleasant (in my opinion). Yet several authors have considered them. Furthermore, despite that the EMT of Definition II is conserved, the associated EOS in cosmology turns to be ill defined because it diverges as I will show in a moment.

Specific $f(R)$ models

Given a specific $f(R)$, we integrate the differential equations forward from past to future with suitable "initial conditions". We have considered three specific $f(R)$ models which have become very popular in the literature

- Miranda *et. al.* model (PRL **102**, 221101, 2009)

$$f(R)_{\mathrm{MJW}} = R - \beta R_* \ln\left(1 + \frac{R}{R_*}\right) \quad . \tag{16}$$

  We used $\beta = 2$ and $R_* = H_0^2$.

- Starobinsky model (JETP Lett. **86**, 157 2007)

$$f(R)_{\mathrm{St}} = R + \lambda R_S \left[\left(1 + \frac{R^2}{R_S^2}\right)^{-q} - 1\right] \quad . \tag{17}$$

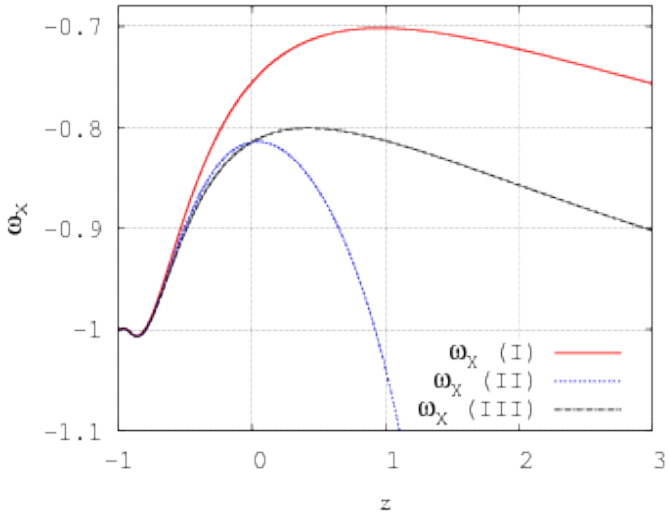
  We take $q = 2$ and $\lambda = 1$, $R_S \approx 4.17 H_0^2$.

- Hu & Sawicky model (PRD **76**, 064004, 2007)

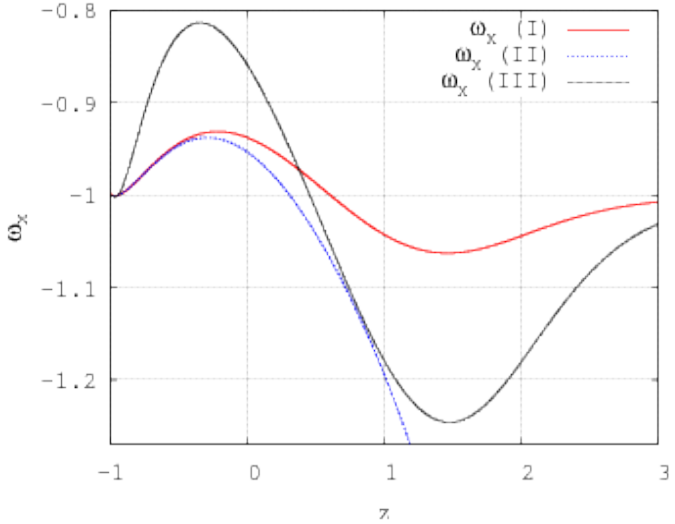
$$f(R)_{HS} = R - m^2 \frac{c_1 (R/m^2)^n}{c_2 (R/m^2)^n + 1} \quad . \tag{18}$$

  We take $n = 4$, $m^2 \approx 0.24 H_0^2$, $c_1 \approx 1.25 \times 10^{-3}$ and $c_2 \approx 6.56 \times 10^{-5}$.

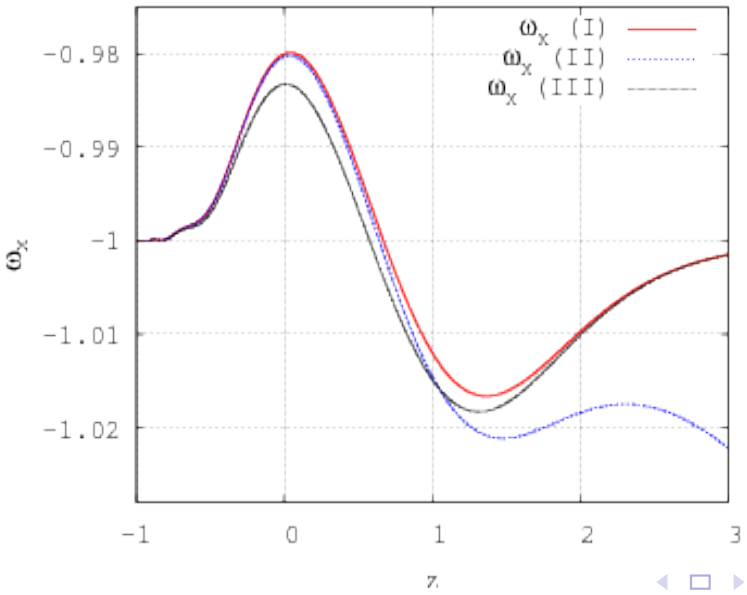
$f(R)_{MJW}$

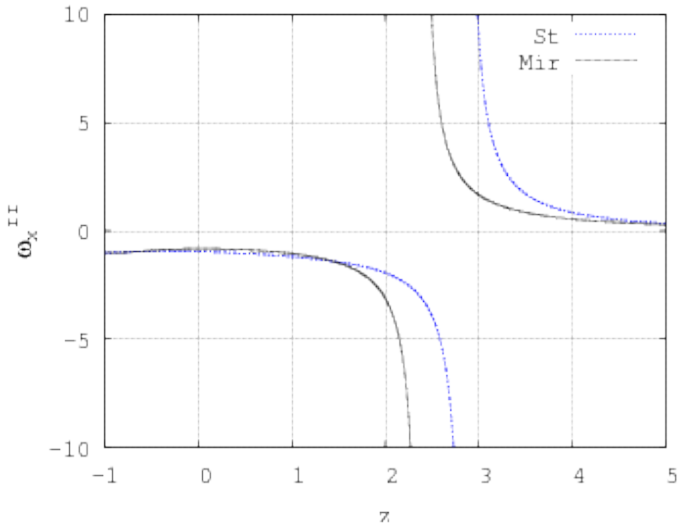


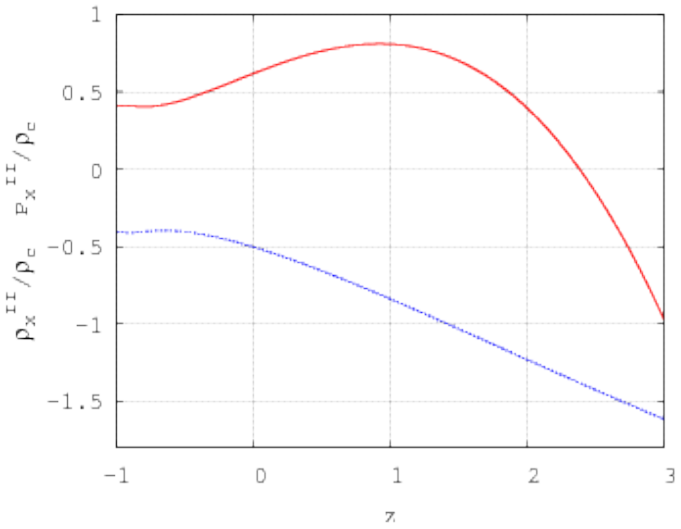
$f(R)_{St}$

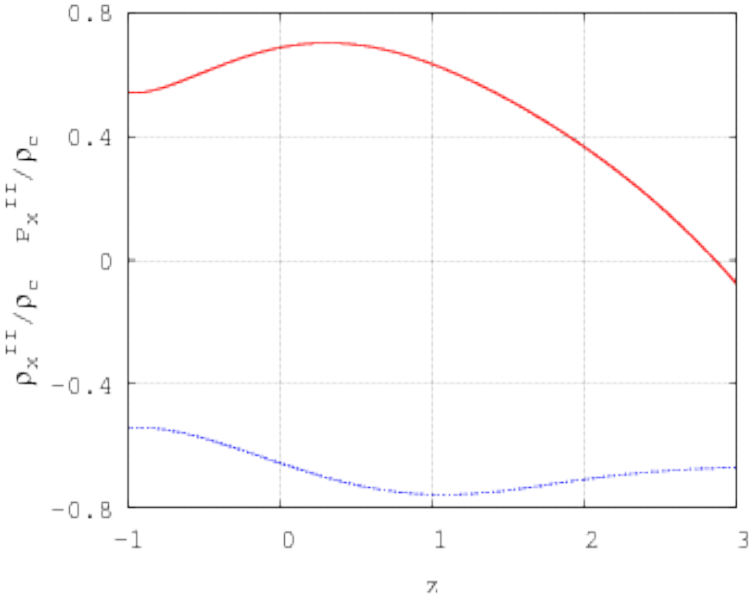


$f(R)_{HS}$




St
Mir
ω_x^{II}
z

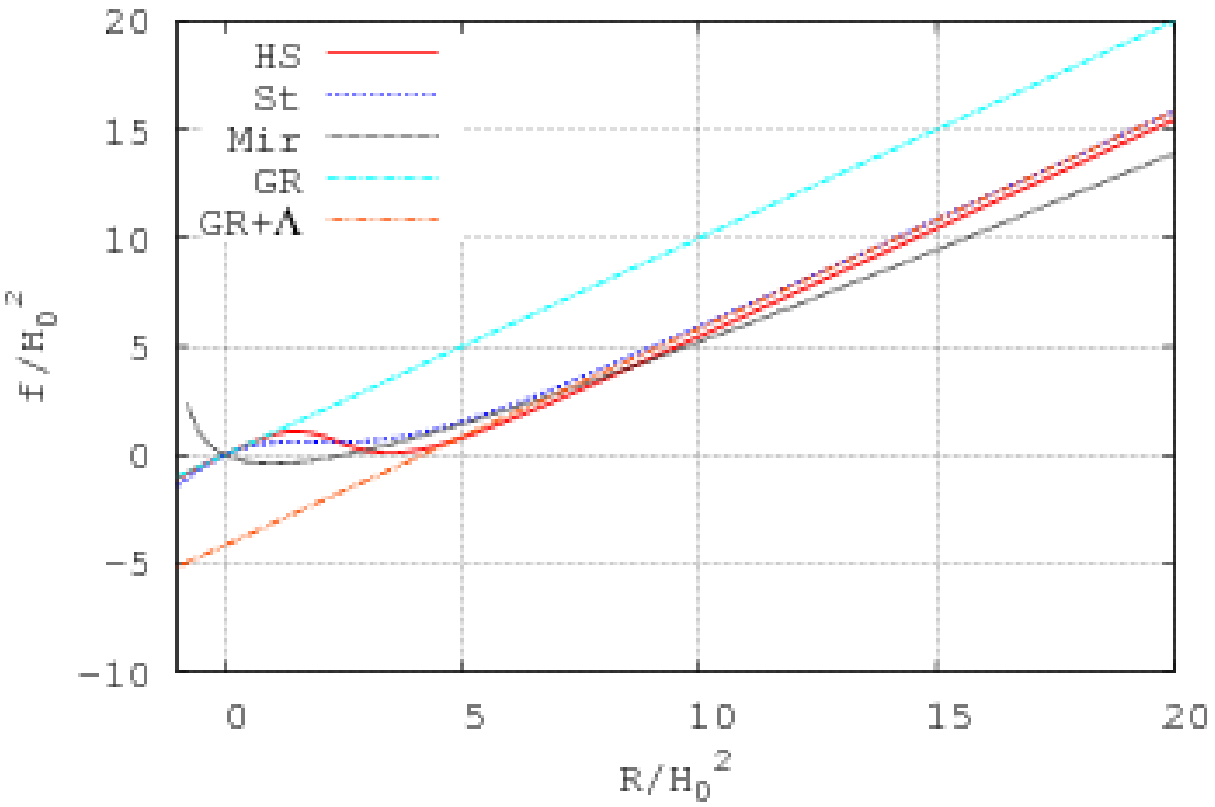

p_x^{II}/ρ_c
ρ_x^{II}/ρ_c
z


p_x^{II}/ρ_c
ρ_x^{II}/ρ_c
z

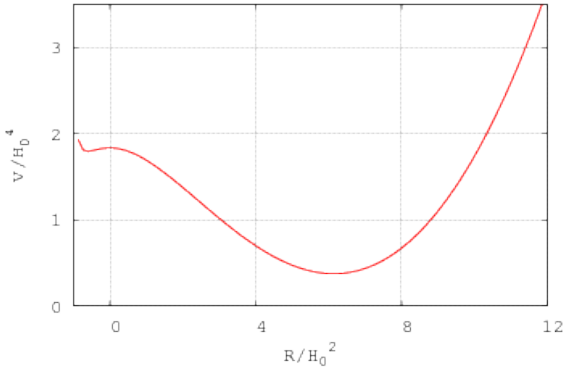
Conclusions

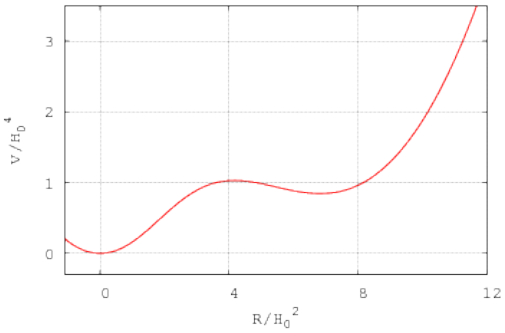
- $f(R)$ theories are alternative theories of gravity that can produce an accelerated expansion of the universe "without" the introduction of Λ. Some specific $f(R)$ models can pass several gravitational tests (e.g. the Solar System tests). They have some predictions different from GR, but I don't have time to discuss them here.
- However, in my opinion they introduce more troubles than solutions. There is no fundamental principle that allows to single out one function $f(R)$. Simplicity favors: $f(R) = R - 2\Lambda$ (i.e. GR+ Λ).
- The EOS associated with $f(R)$ as geometric dark energy is not canonical. We suggest that among the three EOS used in literature ONLY one has the nicest features: Definition I ($A = B = 1$): it arises from a conserved EMT and it is well behaved, unlike the other possibilities. It is important then to agree which EOS will be compared with observations in view of the forthcoming cosmological missions (e.g BigBOSS–DESI–, EUCLID, PanSTARR, WFIRST, etc.) that will determine the EOS of DE with a better precision.

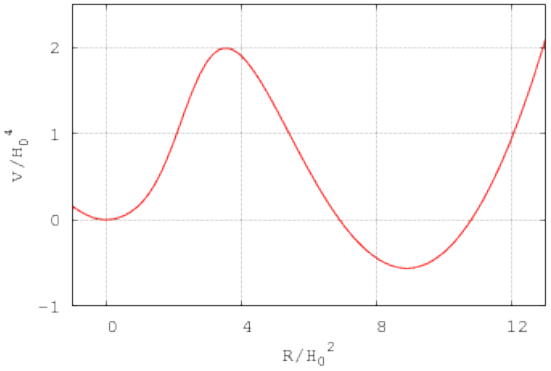
$f(R)$ Models

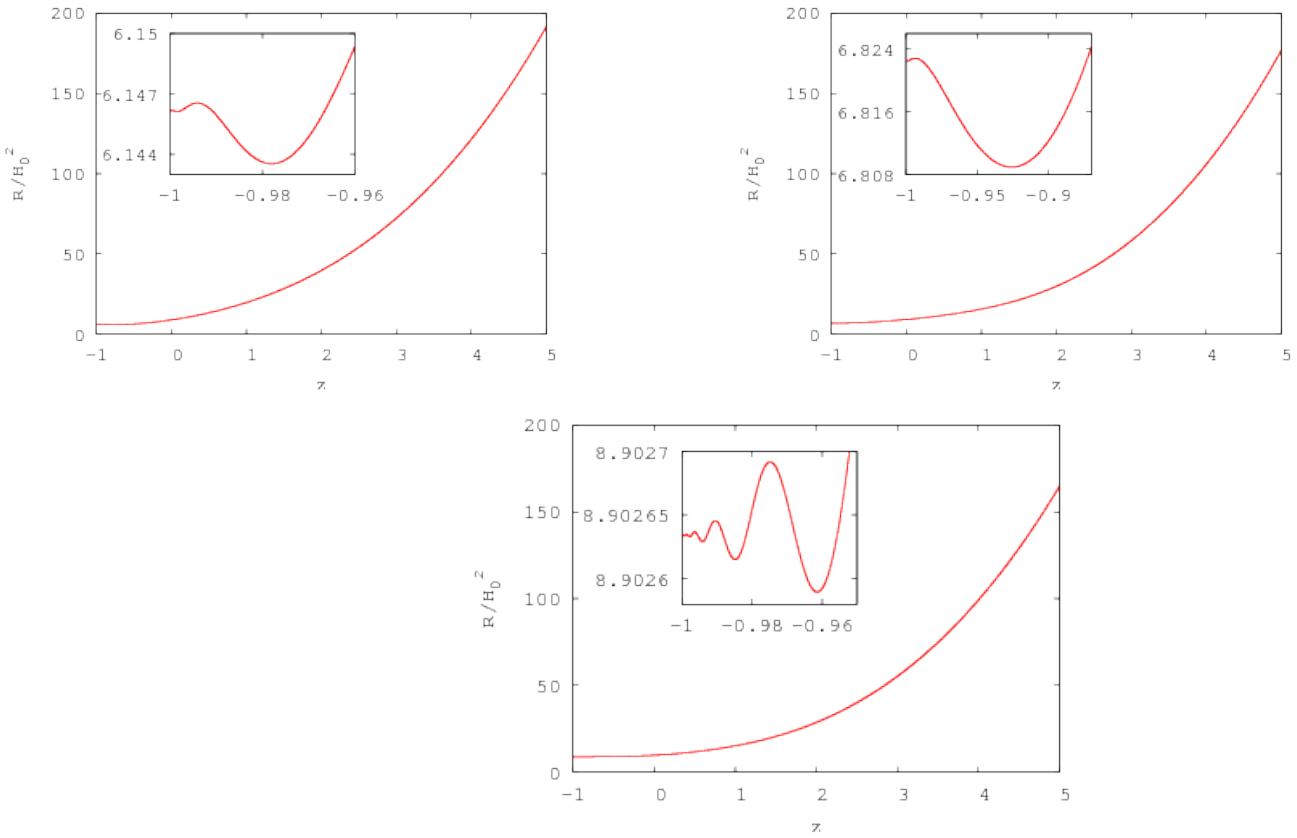


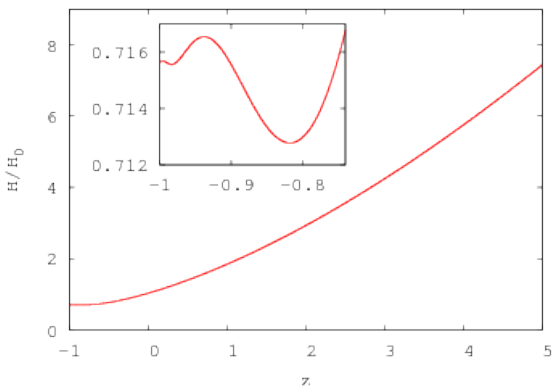
Potentials $V(R) = -Rf(R)/3 + \int^R f(x)dx$ such that $V'(R) = \frac{1}{3}(2f - Rf_R)$. At the extrema of $V(R)$ (notably at the global minimum) the de Sitter "point" is reached where the models behave as a GR plus $\Lambda_{\rm eff} = R_1/4$, where $V'(R_1) = 0$. The specific cosmological models interpolate between a large R (at early time) and near the nontrivial minimum $R \neq 0$ at present time.

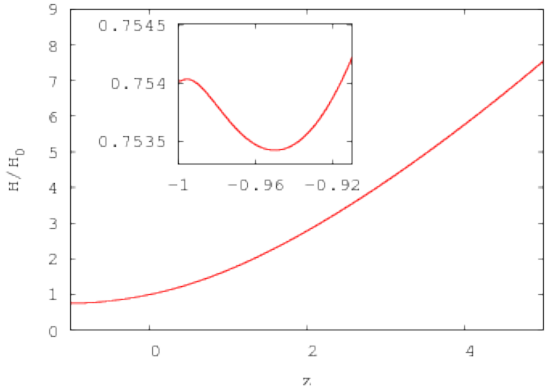







200
150
100
50
0
6.15
6.147
6.144
-1
-0.98
-0.96
R/H0^2
z
-1
0
1
2
3
4
5
6.824
6.816
6.808
-0.95
-0.9
8.9027
8.90265
8.9026


H/H0
z
0.716
0.714
0.712
-1
-0.9
-0.8


H/H0
z
0.7545
0.754
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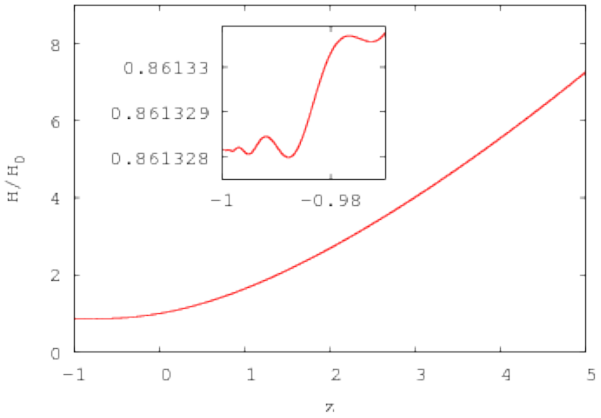
-1
-0.96
-0.92


H/H0
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0.86133
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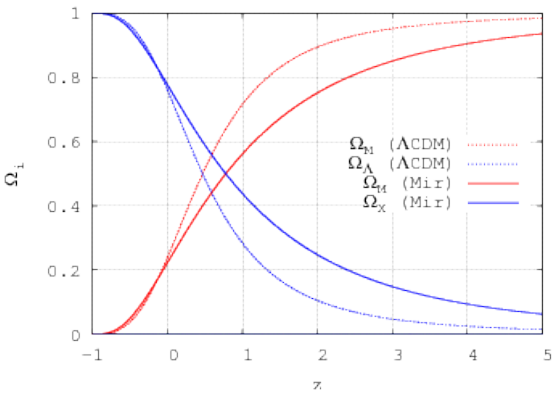
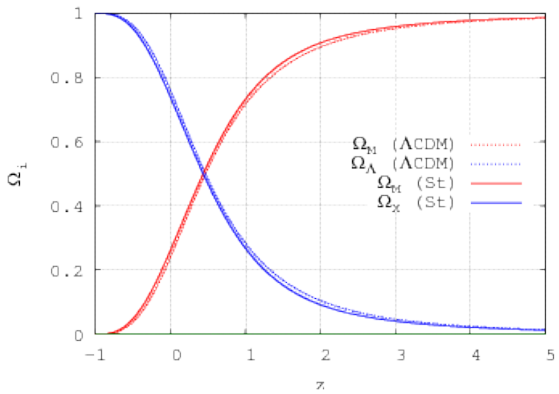
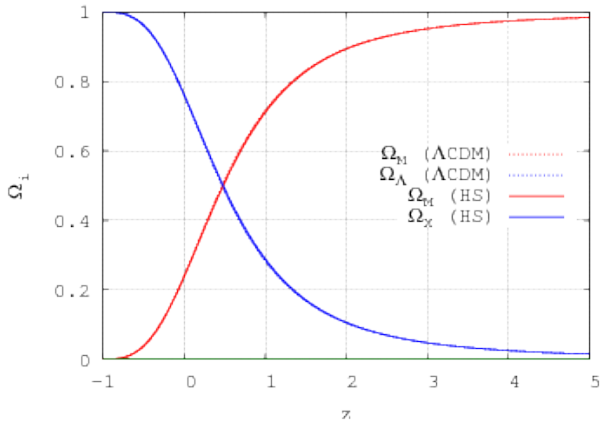
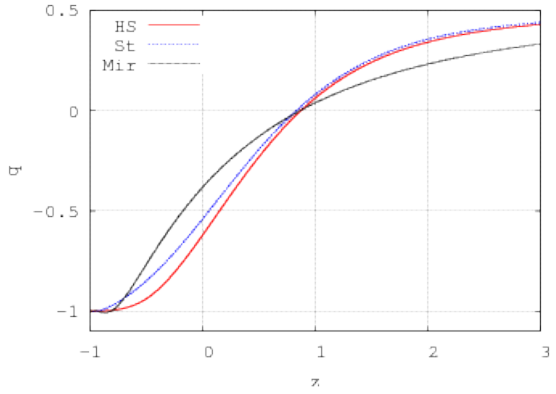

Ω_i
z
Ω_M (ΛCDM)
Ω_Λ (ΛCDM)
Ω_M (Mir)
Ω_X (Mir)

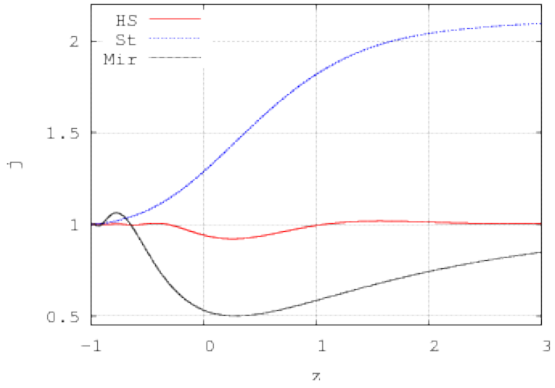

Ω_i
z
Ω_M (ΛCDM)
Ω_Λ (ΛCDM)
Ω_M (St)
Ω_X (St)


Ω_i
z
Ω_M (ΛCDM)
Ω_Λ (ΛCDM)
Ω_M (HS)
Ω_X (HS)

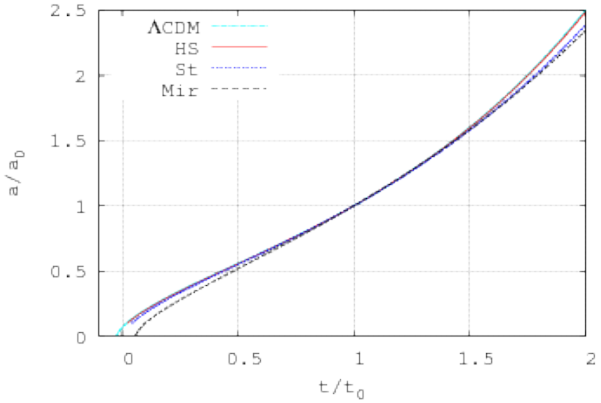
Deceleration parameter: $q := -\frac{\ddot{a}}{aH^2} = -\frac{H^2+\dot{H}}{H^2} = 1 - \frac{R}{6H^2}$

Jerk: $j := \frac{\dddot{a}}{aH^3} = \frac{\dot{R}}{6H^3} - \frac{\dot{H}}{H^2} + 1 = \frac{\dot{R}}{6H^3} + q + 2$

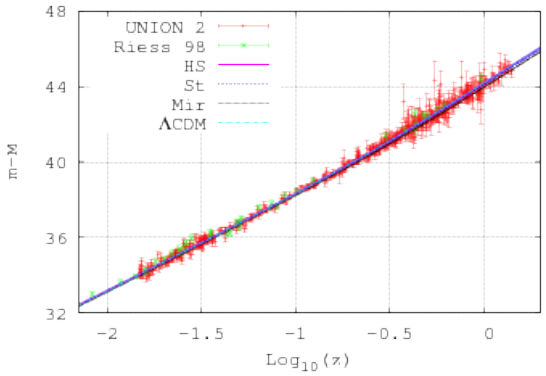


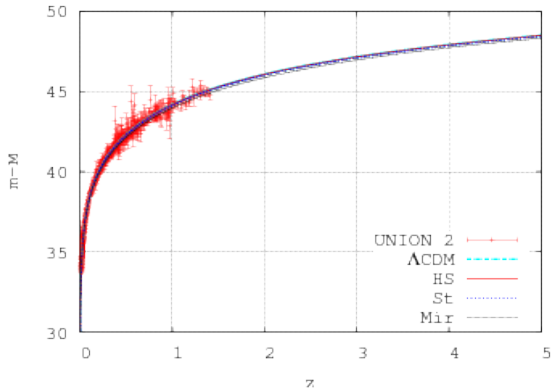


The age of the Universe is $H_0^{-1} \approx 9.78h^{-1} \times 10^9\mathrm{y} \sim 13.97 \times 10^9\mathrm{y}$ (with $h = 0.7$)

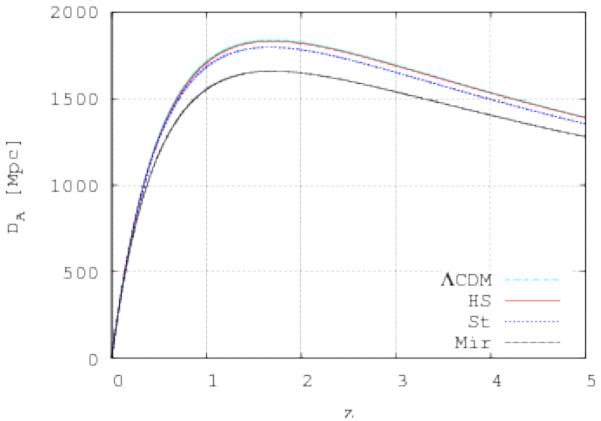


Luminosity distance and SNIa data confrontation ($k = 0$): $d_L^{\rm flat} = \frac{\zeta(\bar{a})}{\bar{a}}$, where $\zeta = c\, H_0^{-1} \int_{\bar{a}}^{1} \frac{d\bar{a}^*}{\bar{a}^{*\,2} \bar{H}(\bar{a}^*)}$. We have also the modulus distance $\mu := m - M = 5\log_{10}(d_L^{\rm flat}/{\rm Mpc}) + 25$.





Diameter angular distance $D_A^{\rm flat} = \bar{a}\zeta(\bar{a})$



- Among the alternative theories of gravity proposed to solve this "crisis" we briefly analyzed $f(R)$ gravity, which has been one of the most popular proposals in recent years and discussed some of its drawbacks and merits.

- Among the alternative theories of gravity proposed to solve this "crisis" we briefly analyzed $f(R)$ gravity, which has been one of the most popular proposals in recent years and discussed some of its drawbacks and merits.
- We also proposed a much more robust approach to treat them and proved its usefulness in the context of cosmology and compact extended bodies.

Equation of State (EOS)

In the ΛCDM model the equation of state $\omega_\Lambda = p_\Lambda/\rho_\Lambda = -1$. We shall define an EOS for the modified gravity contribution given by (for $f(R) \neq R$)

$$\omega_X = \frac{p_X}{\rho_X} \tag{19}$$

where ρ_X is defined from the modified Friedmann equation, so that it reads $H^2 = \frac{\kappa}{3}\left(\rho + \rho_X\right)$, which leads to

$$\rho_X = \frac{1}{\kappa f_R}\left\{\frac{1}{2}\left(f_R R - f\right) - 3 f_{RR} H \dot{R} + \kappa\rho\left(1 - f_R\right)\right\} , \tag{20}$$

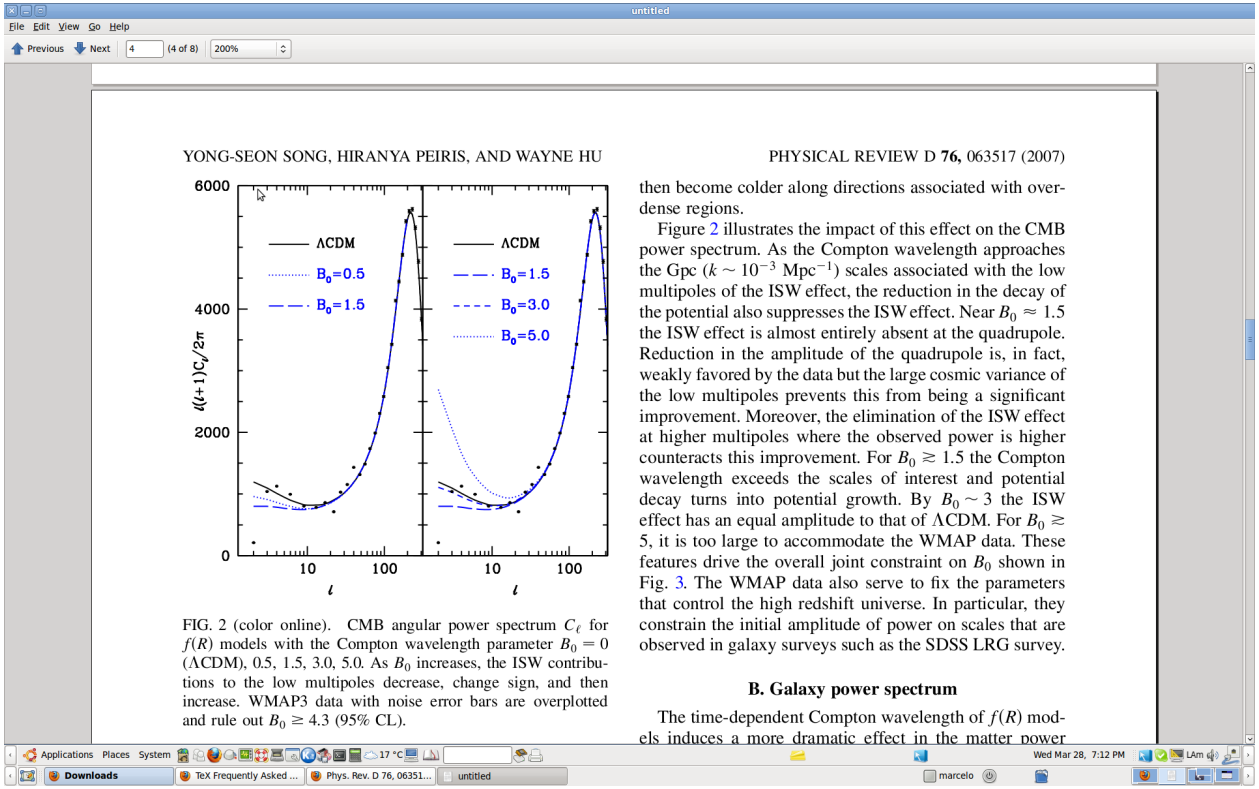
In a similar way we define ρ_X, so that the dynamic equation for H reads

$$\dot{H} + H^2 = -\frac{\kappa}{6}\left\{\rho + \rho_X + 3\left(p_{\rm rad} + p_X\right)\right\} \quad . \tag{21}$$

From this latter, we obtain

$$p_X = \quad -\frac{2}{\kappa f_R}\left\{f_{RR} H \dot{R} + \frac{f}{6} - \frac{k}{3}\rho\right\} - \frac{1}{3}\left(\rho + \rho_X + 3 p_{\rm rad}\right) \ , \tag{22}$$

$f(R)$ Models



YONG-SEON SONG, HIRANYA PEIRIS, AND WAYNE HU PHYSICAL REVIEW D **76,** 063517 (2007)

FIG. 2 (color online). CMB angular power spectrum C_ℓ for $f(R)$ models with the Compton wavelength parameter $B_0 = 0$ (ΛCDM), 0.5, 1.5, 3.0, 5.0. As B_0 increases, the ISW contributions to the low multipoles decrease, change sign, and then increase. WMAP3 data with noise error bars are overplotted and rule out $B_0 \geq 4.3$ (95% CL).

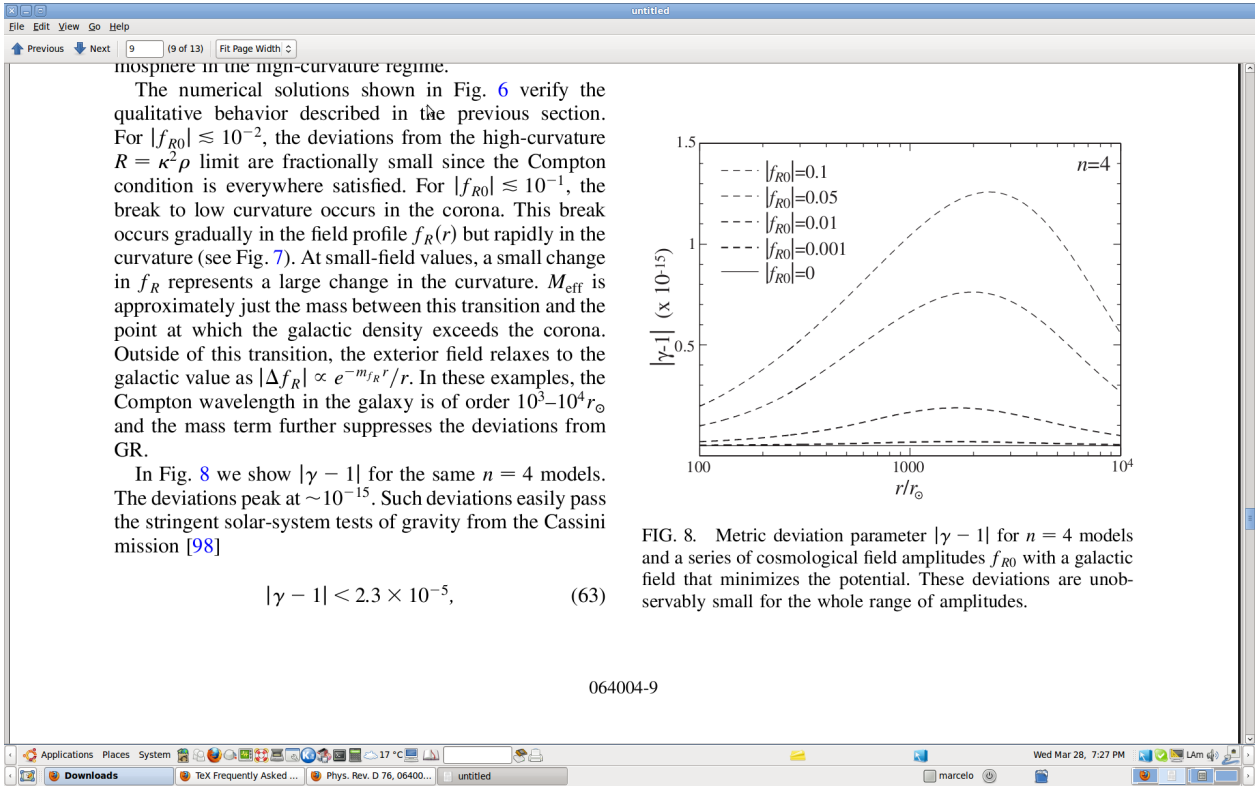
then become colder along directions associated with overdense regions.

Figure 2 illustrates the impact of this effect on the CMB power spectrum. As the Compton wavelength approaches the Gpc ($k \sim 10^{-3}$ Mpc^{-1}) scales associated with the low multipoles of the ISW effect, the reduction in the decay of the potential also suppresses the ISW effect. Near $B_0 \approx 1.5$ the ISW effect is almost entirely absent at the quadrupole. Reduction in the amplitude of the quadrupole is, in fact, weakly favored by the data but the large cosmic variance of the low multipoles prevents this from being a significant improvement. Moreover, the elimination of the ISW effect at higher multipoles where the observed power is higher counteracts this improvement. For $B_0 \gtrsim 1.5$ the Compton wavelength exceeds the scales of interest and potential decay turns into potential growth. By $B_0 \sim 3$ the ISW effect has an equal amplitude to that of ΛCDM. For $B_0 \gtrsim 5$, it is too large to accommodate the WMAP data. These features drive the overall joint constraint on B_0 shown in Fig. 3. The WMAP data also serve to fix the parameters that control the high redshift universe. In particular, they constrain the initial amplitude of power on scales that are observed in galaxy surveys such as the SDSS LRG survey.

B. Galaxy power spectrum

The time-dependent Compton wavelength of $f(R)$ models induces a more dramatic effect in the matter power

$f(R)$ Models



mosphere in the high-curvature regime.

The numerical solutions shown in Fig. 6 verify the qualitative behavior described in the previous section. For $|f_{R0}| \lesssim 10^{-2}$, the deviations from the high-curvature $R = \kappa^2\rho$ limit are fractionally small since the Compton condition is everywhere satisfied. For $|f_{R0}| \lesssim 10^{-1}$, the break to low curvature occurs in the corona. This break occurs gradually in the field profile $f_R(r)$ but rapidly in the curvature (see Fig. 7). At small-field values, a small change in f_R represents a large change in the curvature. $M_{\rm eff}$ is approximately just the mass between this transition and the point at which the galactic density exceeds the corona. Outside of this transition, the exterior field relaxes to the galactic value as $|\Delta f_R| \propto e^{-m_{f_R} r}/r$. In these examples, the Compton wavelength in the galaxy is of order 10^3–$10^4 r_\odot$ and the mass term further suppresses the deviations from GR.

In Fig. 8 we show $|\gamma - 1|$ for the same $n = 4$ models. The deviations peak at $\sim 10^{-15}$. Such deviations easily pass the stringent solar-system tests of gravity from the Cassini mission [98]

$$|\gamma - 1| < 2.3 \times 10^{-5}, \qquad (63)$$

FIG. 8. Metric deviation parameter $|\gamma - 1|$ for $n = 4$ models and a series of cosmological field amplitudes f_{R0} with a galactic field that minimizes the potential. These deviations are unobservably small for the whole range of amplitudes.

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Many more $f(R)$ models have been analyzed. In fact **Amendola et al.** (PRL **98**,131302,2007; PRD **75**,083504) developed some general criteria to asses if a model is not cosmologically viable. According to these authors, most of the following models whether are unable to produce a late time accelerated expansion or they behave as radiation in the time where matter dominations is supposed to take place (just before the era of "dark energy" domination). Therefore such models would not allow structure formation:

$$f(R) = R - \frac{\mu^4}{R} , \tag{23}$$

$$f(R) = R - \frac{\mu_1^4}{R} + \mu_2^4 R , \tag{24}$$

$$f(R) = \alpha R^{-n} , \tag{25}$$

$$f(R) = R + \alpha R^{-n} , \text{ (possibly viable for } \alpha < 0, n \approx 1) \tag{26}$$

$$f(R) = R^p e^{qR} , \tag{27}$$

$$f(R) = R^p (\log \alpha R)^2 , \text{ (might succed for } p = 1, q > 0, q \neq 1) \tag{28}$$

$$f(R) = R^p e^{q/R} , \tag{29}$$

$$f(R) = R + \alpha R^2 , \tag{30}$$

As one can appreciate, a large variety of models have been analyzed, but people have come up with new proposals. Like those I showed before. However, such modesl might have other problems.

- In fact it was only recently that several authors have tried to construct relativistic extended objects in the framework of $f(R)$ gravity.
- In particular **Kobayashi and Maeda** (PRD **78**, 064019, 2008; PRD **79**, 024009, 2009) using the Starobinsky model $f(R) = R - \lambda R_* \left\{1 - \left[1 + (R/R_*)^2\right]^{-\beta}\right\}$ shown that such objects cannot be constructed because a curvature singularity developed within the object.
- Later **Babichev and Langlois** (PRD **80**, 121501(R) 2009; gr-qc/0911.1297, 2010) reanalyzed the issue and concluded that KM results was a consequence of the use of an incompressible fluid, and that using a more realistic EOS (polytropes) such singularities were not found.
- However, **Upadhye and Hu** (PRD **80**, 064002,2009) found that relativistic extended objects can indeed be constructed, but that the absence of singularities got nothing to do with the EOS, but rather with a "chameleon mechanism".

- The common feature of the aforementioned works is that authors used the above mapping to constuct such objects.
- Under such mapping the Ricci scalar has a behavior of the sort $R \sim 1/(\chi - \chi_0)$ where $\chi := \partial_R f$ and $\chi_0 = \mathrm{const}$.
- The key point is to determine if the dynamics of χ leads it or not to the value $\chi = \chi_0$ within the spacetime generated by the relativistic object.
- Irrespective of the different results and confusing explanations obtained by those works we argue that their conclusions are rather questionable due to the fact that the above scalar-field variables are ill defined. To be more specific, the scalar-field potential used to study the dynamics of χ is not single valued and possesses pathological features. Since similar kind of singularities were also found in the cosmological setting (Frolov, PRL **101**, 061103, 2008), it is then worrisome that the ill-defined potential play such a crucial role in those analyses (several authors have already criticized the use of such potentials).

We consider the following metric that allows us to describe SSS

$$ds^2 = -n(r)dt^2 + m(r)dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) . \tag{31}$$

The field Eqs. then read

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The field Eqs. then read

$$R'' = \frac{1}{3f_{RR}}\left[m(\kappa T + 2f - Rf_R) - 3f_{RRR}R'^2\right] + \left(\frac{m'}{2m} - \frac{n'}{2n} - \frac{2}{r}\right)R' . \tag{32}$$

(where $' := d/dr$). From the $t - t$, $r - r$, and $\theta - \theta$ of field Eqs. and after several non-trivial manipulations we found

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$$\begin{aligned} m' &= \frac{m}{r(2f_R + rR'f_{RR})}\Bigg\{2f_R(1-m) - 2mr^2\kappa T^t{}_t + \frac{mr^2}{3}(Rf_R + f + 2\kappa T) + \frac{rR'f_{RR}}{f_R}\Big[\frac{mr^2}{3}(2Rf_R - f + \kappa T) \\ &\quad -\kappa mr^2(T^t{}_t + T^r{}_r) + 2(1-m)f_R + 2rR'f_{RR}\Big]\Bigg\} , \end{aligned} \tag{33}$$

$$n' = \frac{n}{r(2f_R + rR'f_{RR})}\left[mr^2(f - Rf_R + 2\kappa T^r{}_r) + 2f_R(m-1) - 4rR'f_{RR}\right] , \tag{34}$$

$$n'' = \frac{2nm}{f_R}\left[\kappa T^\theta{}_\theta - \frac{1}{6}(Rf_R + f + 2\kappa T) + \frac{R'}{rm}f_{RR}\right] + \frac{n}{2r}\left[2\left(\frac{m'}{m} - \frac{n'}{n}\right) + \frac{rn'}{n}\left(\frac{m'}{m} + \frac{n'}{n}\right)\right] . \tag{35}$$

Remarks of the above system of ODE's:

- Notice that Eqs. for n' and n'' are not independent. In fact, one has the freedom of using one or the other. Nevertheless, we have used both to check the consistency of our equations and the numerical code (solutions).
- Now, from the usual expression of R in terms of the Christoffel symbols one obtains,

$$R = \frac{1}{2r^2n^2m^2}\Big[4n^2m(m-1) + rnm'(4n+rn') - 2rnm(2n'+rn'') + r^2mn'^2\Big]\,. \tag{36}$$

  As one can check by a direct calculation, that using the Eqs. for m', n', n'' in the above Eq., one finds an identity $R \equiv R$. This result confirms two things: 1) Our Eqs. are consistent and no elementary mistake was made in their derivation; 2) The previous expression for R does not provide any further information.
- When defining the first order variables $Q_n = n'$ and $Q_R := R'$, the above system of ODE's have the form $dy^i/dr = \mathcal{F}^i(r, y^i)$ where $y^i = (m, n, Q_n, R, Q_R)$ and therefore can be solved numerically. As far as we are aware, such a system has not been considered previously. These equations can be used to tackle several aspects of SSS spacetimes in $f(R)$ gravity.

- We observe that for $f(R) = R$ our system of ODE's reduce to the well known equations of GR for SSS spacetimes.

- Like in the general case, our system of ODE's in vacuum has the exact de Sitter solution $n(r) = m(r)^{-1} = 1 - \Lambda_{\rm eff} r^2/3$, $R = R_1 = \mathrm{const.}$ with $\Lambda_{\rm eff} = R_1/4$ and $R_1 = 2f(R_1)/f_R(R_1)$.

- We also need the matter equations $\nabla^a T_{ab} = 0$. So for $T_{ab} = (\rho + p)u_a u_b + g_{ab} p$, we get

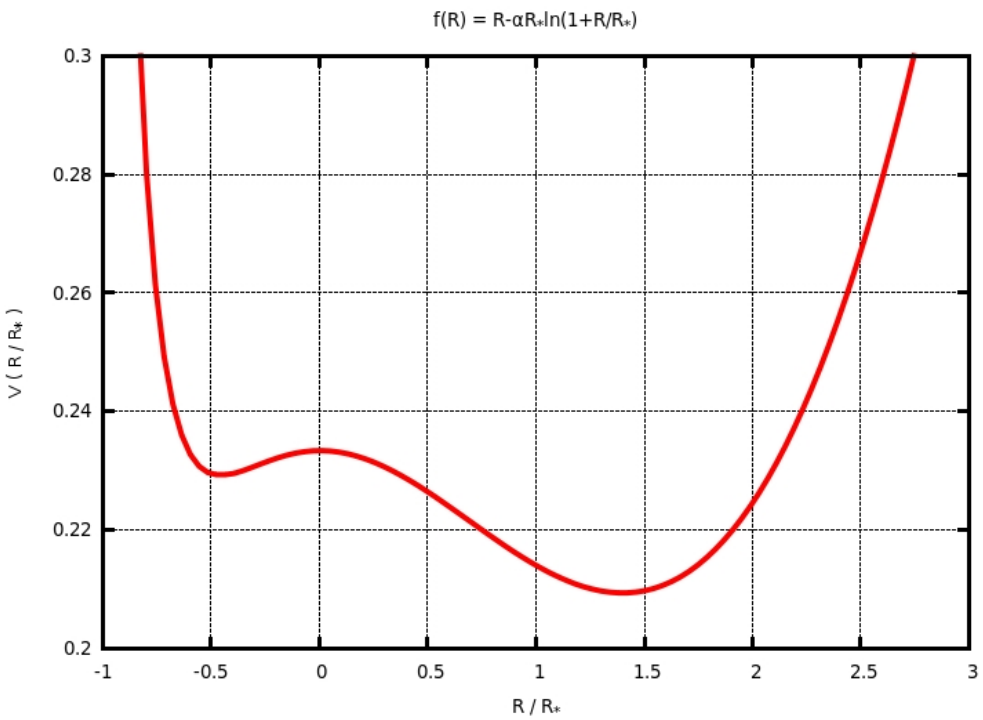
$$p' = -(\rho + p)n'/2n \tag{37}$$

This is the modified Tolman-Oppenheimer-Volkoff equation of hydrostatic equilibrium which is to be complemented by an EOS.

Numerical Results

- For simplicity we shall assume an incompressible fluid ($\rho = const.$)
- We integrate the equations numerically outwards from the origin $r = 0$ and impose regularity conditions at $r = 0$. We fix $p(0)$ (the central pressure) to a given value (this fixes one "star" configuration).
- We obtain $R(0)$ by a shooting method so that asymptotically the solution matches the de Sitter solution $R = R_1 = const.$ where R_1 is a critical point of the "potential" $V(R) = -Rf(R)/3 + \int^R f(x)dx$. That is, R_1 is a point where $dV(R)/dR = (2f - f_R R)/3$ vanishes.
- This potential is radically different from the scalar-field potential that arises under the STT map. Furthermore, $V(R)$ is as well defined as the function $f(R)$ itself.

- We used first the model $f(R) = R - \alpha R_* \ln(1 + R/R_*)$ (α, R_* are positive constants; R_* sets the scale) proposed by **Miranda et. al.** (PRL **102**, 221101, 2009). Warning: for this model f_R is not positive definite in general (only if $\alpha < 1 + R/R_*$) but $f_{RR} > 0$. Note also that $f(R)$ is only well defined for $0 < 1 + R/R_*$. A priori there is no guarantee that solutions for R exist satisfying such conditions.
- Those authors mapped the theory to the STT counterpart. However, unlike the Starobinsky model (see below), in this case the resulting scalar-field potential turns to be single-valued. This is why we take it in order to compare (calibrate) directly with our method.
- They did not find any singularity within the object.
- Under our approach we associate to this $f(R)$ the potential $V(R) = \frac{R_*^2}{6}\left\{(1+\tilde{R})\left[\tilde{R} + (6\alpha - 1)\right] - 2\alpha(3 + 2\tilde{R})\ln\left[1 + \tilde{R}\right]\right\}$, where $\tilde{R} = R/R_*$. For $\alpha = 1.2$ (the value that Miranda et al. assumed) this potential has several critical points



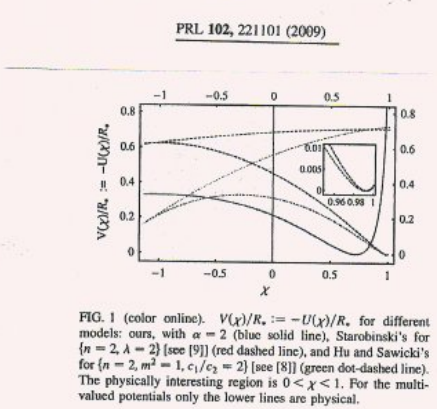
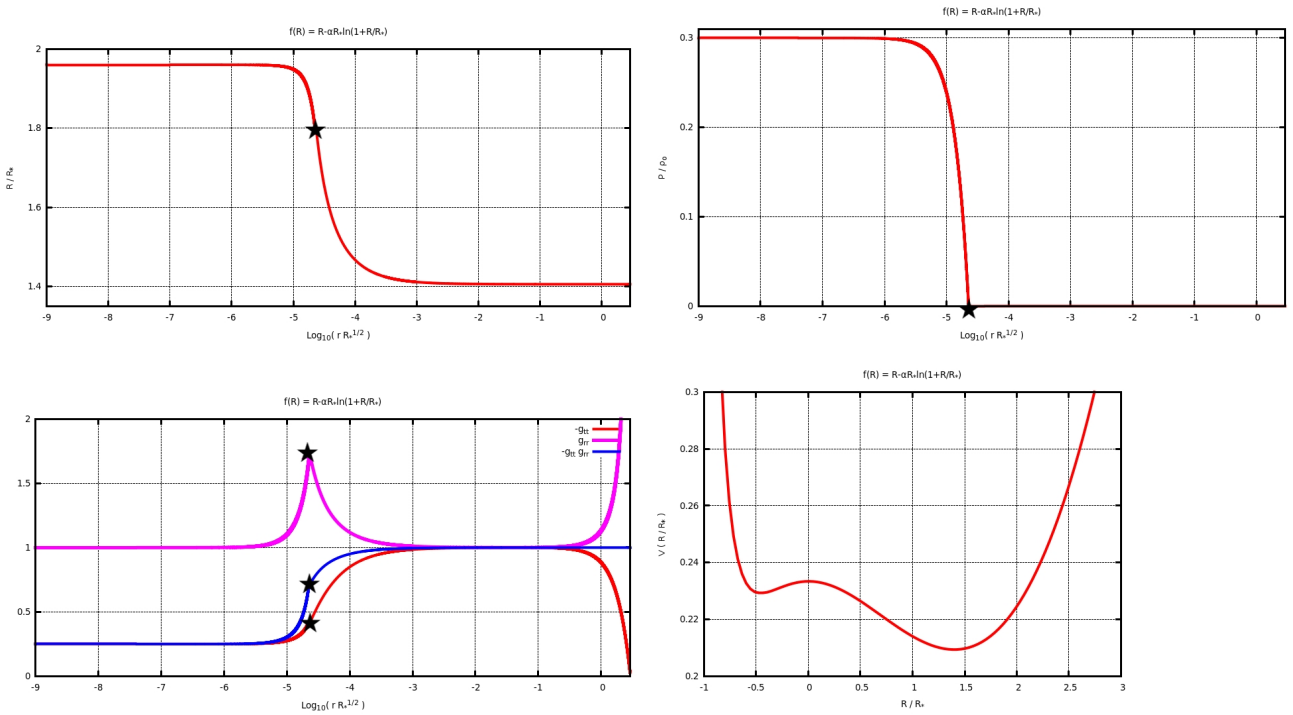
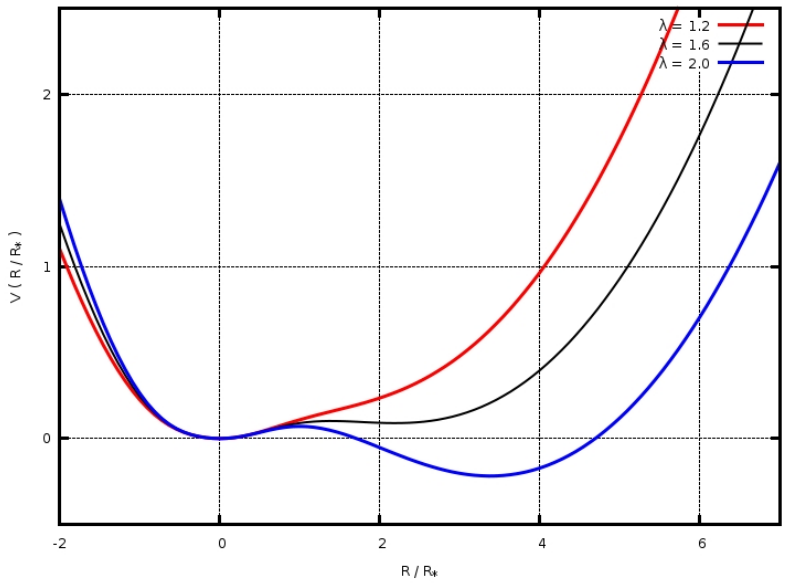

PRL 102, 221101 (2009)

FIG. 1 (color online). $V(\chi)/R_* := -U(\chi)/R_*$ for different models: ours, with $\alpha = 2$ (blue solid line), Starobinski's for $\{n = 2, \lambda = 2\}$ [see [9]] (red dashed line), and Hu and Sawicki's for $\{n = 2, m^2 = 1, c_1/c_2 = 2\}$ [see [8]] (green dot-dashed line). The physically interesting region is $0 < \chi < 1$. For the multi-valued potentials only the lower lines are physical.

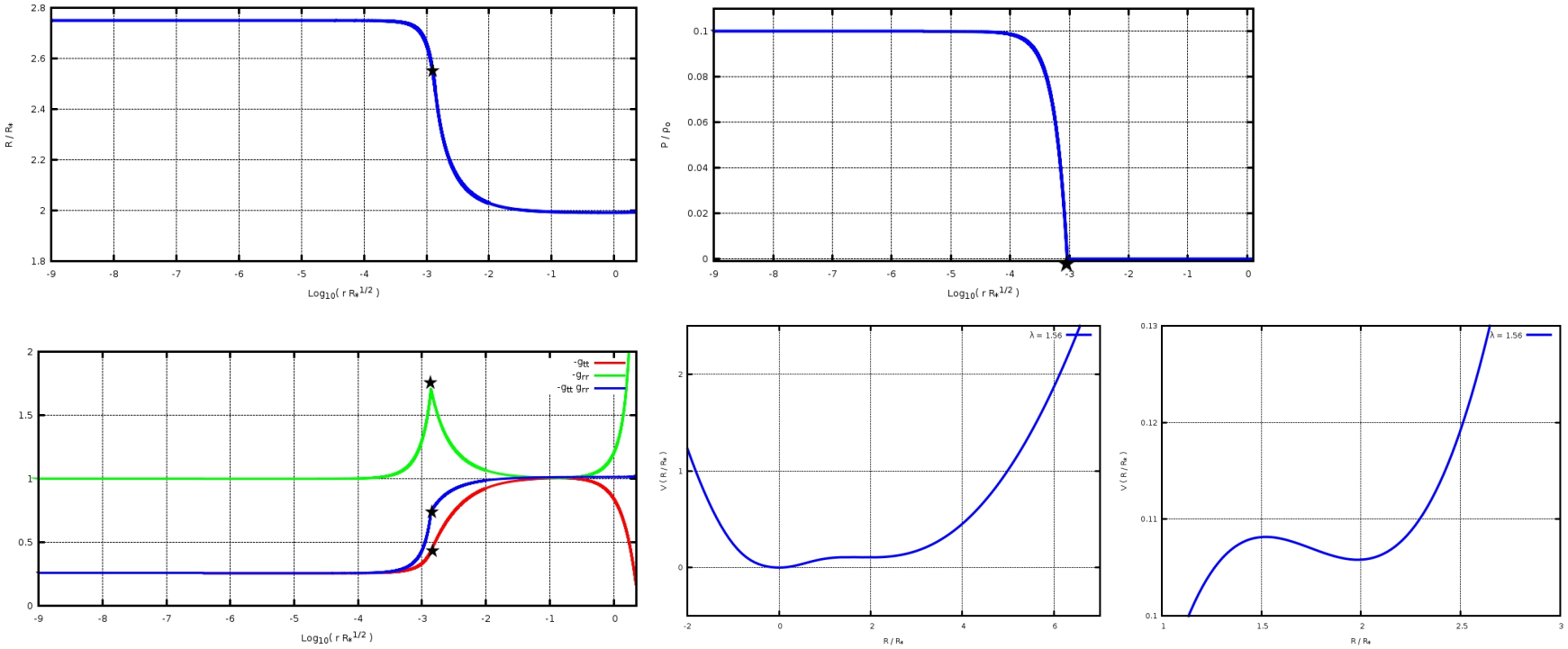
We have used our approach and found no singularities whatsoever in compact objects (like in Miranda et al.)



- We then used the Starobinsky model $f(R) = R - \lambda R_* \left\{1 - \left[1 + (R/R_*)^2\right]^{-\beta}\right\}$ with $\beta = 1$. The controversy about the existence of extended relativistic objects (or absence thereof) was originated using this model. As we saw, the STT approach gives rise to multivalued potentials.
- With our method, the potential is given by ($\tilde{R} = R/R_*$)
$$V(R) = \frac{R_*^2}{3}\left\{\frac{\tilde{R}}{2}\left[\tilde{R} - 4\lambda - 2\lambda\left(1 + \tilde{R}^2\right)^{-1}\right] - 3\lambda \arctan\left(\tilde{R}\right)\right\}.$$
- This potential has a rich structure depending on the value of λ.



For the value $\lambda = 1.56$ we found the following solutions using a shooting method aiming to the local minimum. No singularities whatsoever where found.



Discussion:

Although we have not find singularities in the spacetime generated by this compact objects, there is a caveat in the above construction:

In almost all $f(R)$ models the dimension parameters like R_*, which settles the scale $\sim D^{-2}$, are chosen such that $R_*/G_0 \sim \tilde{\Lambda} \sim 10^{-29}\mathrm{g\,cm^{-3}}$ ($\tilde{\Lambda} = \Lambda/G_0$). On the other hand, if one wants to build a realistic neutron star, the typical densities at the center are $\rho(0) \sim \rho_{\mathrm{nuc}} \sim 10^{14}\mathrm{g\,cm^{-3}}$. That is, there are around 43 orders of magnitude between the typical density within a neutron star and the average density of the Universe! This ratio between densities naturally appears in the equations since the parameters which define the specific $f(R)$ theory are of the order of $\tilde{\Lambda}$, while the appropriate dimension within neutron stars is ρ_{nuc}. So in units of ρ_{nuc}, the cosmological constant turns out to be ridiculously small, while in units of $\tilde{\Lambda}$, $\rho(r)$ and $p(r)$ turn out to be ridiculously large within the neutron star.

In other words, the scale of a neutron star is $\sim \mathrm{km}$ while the cosmological scales are $\sim 100\mathrm{Mpc}$. We, like the other authors, have not solved this technical problem...YET, and therefore have constructed "compact" objects which whether are realisitic but then $\tilde{\Lambda}$ is not. Or the opposite. In both cases the objects are compact and relativistic in the sense that $p \sim \rho$ and $G_0\mathcal{M}/\mathcal{R}$ is not far from $4/9$, where $\mathcal{M}$ is an ADM mass defined in asymptotically de Sitter spacetimes.

Solar system tests: weak field limit. Consider static and spherically symmetric perturbations ($|\phi|, |\psi| \ll 1$) around a De Sitter background:

$$ds^2 = -(1 - \phi - \Lambda_{eff} r^2)dt^2 + (1 + \psi - \Lambda_{eff} r^2)dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right) \quad (38)$$

where ϕ, ψ will vanishing asymptotically. Moreover $R = R_1 + \tilde{R}$ where $|\tilde{R}| \ll |R_1|$. Recall $R_1 = 4\Lambda_{eff}$. One can show that outside a spherical body, the solution is

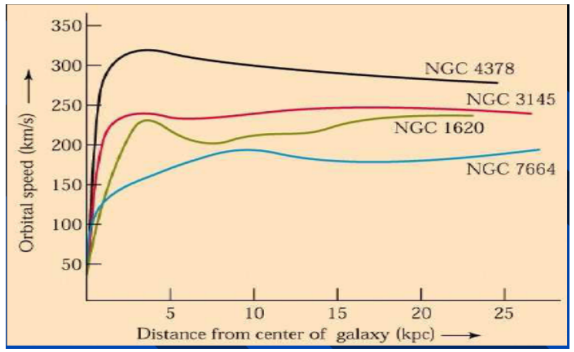
$$\phi = 2M/r \quad (39)$$

$$\psi = M/r \quad (40)$$

$$(41)$$

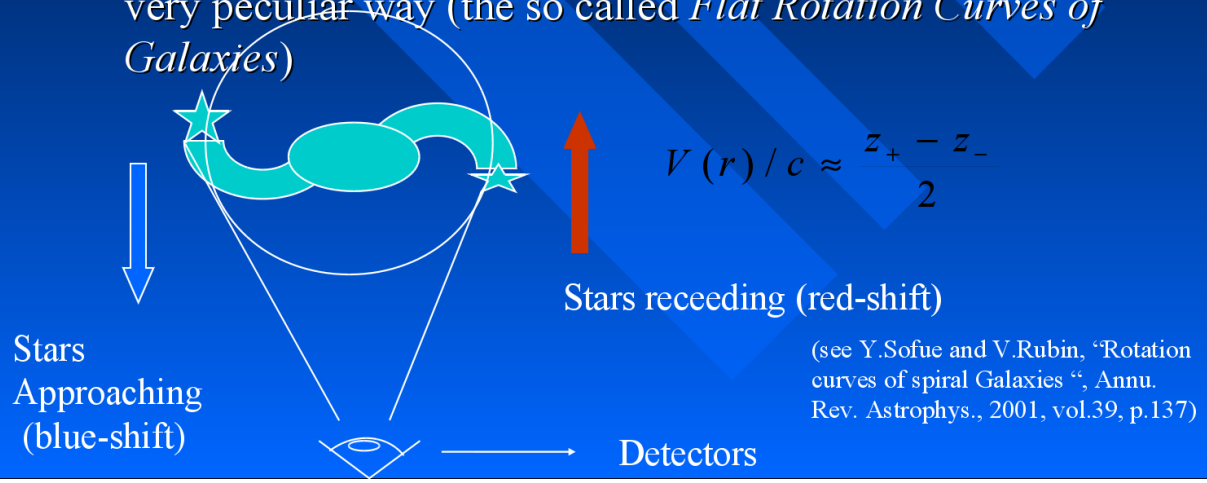
But the post-Newtonian parameter $\gamma = |\psi|/|\phi| = 1/2$ while the measurements indicate $\gamma = 1 \pm a$ where $a \sim 10^{-5}$ (Cassini probe: Bertotti *et al.* Nature **425**, 2003, 474). This result is independent of the particular $f(R)$ model . So in principle $f(R)$ gravity would be ruled out. However, people discover that some $f(R)$ models can develop a *chameleon*: effect that makes modifications to ordinary gravity so that it decays exponentially in regions of "high" density while decay slower in regions of "low" density. So within the Sun, the effect of the scalar R would be negligeable, while in cosmic enviroments would be important.

Later (1970's), observations in spiral galaxies showed that gas and stars revolved around the center of the galaxy in a very peculiar way: their tangential velocity $V(r) \approx const.$ for some $r > r_b \sim 5kpc$ instead of falling off as $V^2 \sim M_{vis}/r$, according to the Newtonian expectations. Here $M_{vis} \sim const. \sim 10^{11} M_\odot$ for $r > r_b$ stands for the total visible mass within the galactic core with radius r_b.



2) Empirical evidence: Since quite a long time (since time of Zwicky) there was evidence that astrophysical matter behaved in a unsual way (dynamics in clusters of galaxies). It seems that it was Zwicky himslef who coined the term "dark matter" to explain the dynamics of some of such systems.

More recently, observations in spiral galaxies showed that gas, and stars revolved around the center of the galaxy in a very peculiar way (the so called *Flat Rotation Curves of Galaxies*)



$$V(r)/c \approx \frac{z_+ - z_-}{2}$$

(see Y.Sofue and V.Rubin, "Rotation curves of spiral Galaxies ", Annu. Rev. Astrophys., 2001, vol.39, p.137)

In order to explain these curves, people started to introduce the idea that around the visible galactic matter there exists a **Dark Matter Halo** which is 10 times more massive than the luminous matter and which produce stronger gravity. Since according to Newtonian expectations $V^2/r \sim M(r)/r^2$, where now $M(r)$ accounts for the total mass including the Dark and visible Matter, one needs then that $M(r) \sim r$ grows linearly for $V(r) \approx const$. Moreover, since $M \sim \int \rho r^2 dr$, the dark matter contribution to the density should behave as $\rho \sim 1/r^2$. So one of the challenges of the Dark Matter supporters is to find a Universal self-gravitating dark matter model whose energy density behave this way.

Rotation curves in spiral galaxies: Assume that the spacetime generated by a galaxy is static and spherically symmetric.

$$ds^2 = -N^2(r)dt^2 + A^2(r)dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{42}$$

Assume that gaz and stars in the galaxy's arms behave as test particles and revolve around the center in circular orbits on the galactic plane. Then the difference of blue shift and red shift from the matter approaching and receeding from us are given by

$$z_D = \frac{z_+ - z_-}{2} = \frac{1}{N}\left(1 - r\partial_r N/N\right)^{-1/2}\left(r\partial_r N/N\right)^{1/2} \tag{43}$$

Usually z_D is identified with the tangential velocity of the revoling matter. This result in theory independent. Usually $N(r)$ is given from the solution of the field equations of the underlying theory. For instance in GR one has $N = (1 - 2GM/r)^{1/2}$ $\approx (1 - GM/r)$. So at first order in GR/r one has $z_D \approx (GR/r)^{1/2}$. However, for $f(R)$ gravity the result might be different. Notably if $f(R)$ is given *a priori*. But one can follow a different strategy like in many Dark Matter models: one can fix $z_D = const.$ then obtain $N(r)$ from Eq.(43) and from the field equations reconstruct $f(R)$. This approach has been followed by several authors. Nevertheless there is no guarantee that such a designer model will satisfy all the remaining observations nor that such a model will serve to reproduce all the rotation curves.