Study of the Schrödinger-Poisson system for applications in Astrophysics and Cosmology.

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Puerto Vallarta, January, 2011.

Outline

1. Schrödinger-Poisson system
   - SP system in Physical space
   - SP system in the Fourier space
2. Numerical Result
   - Equilibrium Configuration
3. Final Comments

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Motivation

The study of scalar fields are very important for both Cosmology and Astrophysics, because the scalar fields are candidates to be the dark matter, which is believed to be responsible for the structure formation and evolution of galaxies. The dynamical study of a massive scalar field can yield information about the features of the dark matter. This study can be done using the Schrödiger-Poisson system (SP). We solve the SP system in 1D for the equilibrium configuration. We analyze the solutions in Physical and Fourier spaces.

SP system in Physical space

The SP system is the weak field version of the Einstein-Klein-Gordon (EKG) system. SP equations in one dimension are represented by

$$i\hbar\frac{\partial\phi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\phi}{\partial x^2} + mc^2 V\phi. \quad (1)$$

$$\frac{d^2V}{dx^2} = 4\pi G\left(\frac{m}{\hbar c}\right)^2\phi^*\phi. \quad (2)$$

where ϕ is the scalar field, m the mass of the field and V the gravitational potential. We will be using units in which $\hbar = c = 1$ and for numerical purposes we define the variables $\phi \to \sqrt{4\pi G}\phi$, $\frac{\partial}{\partial t} \to \frac{1}{m}\frac{\partial}{\partial t}$, $\frac{\partial}{\partial x} \to \frac{1}{m}\frac{\partial}{\partial x}$, for $-L \leq x \leq L$.

Numerical Methods

We solve the Poisson equation numerically (2) using Chebyshev Collocation Spectral Method. We propose

$$V(x_i) = \sum_{n=1}^{M} a_n T_n(x_i). \tag{3}$$

where $T_n(x_i) = \cos(n \cos^{-1}(x_i))$ are the Chebyshev polynomials and $x_i = \cos(i\pi/M)$ are the set of points that we chose to solve the equation (3) to find the coefficients a_n . We substitute (3) into (2) and obtain

$$\sum_{n=1}^{M} a_n \frac{d^2 T_n(x_i)}{dx^2} = \phi^*(x_i)\phi(x_i) \tag{4}$$

The last equation can be written as a linear matrix for the coefficients a_n

$$\mathbf{M}\vec{a} = \vec{b}, \tag{5}$$

where the components of the matrix $\mathbf{M}$ are $T^{''}(x_i)$ (second order derivative of the Chebyshev polynomials), the a components are given by the coefficients a_n which are unknown and the b components are given by the right hand side of the equation (4).

To know the coefficients a_n, we need to solve the system equation (5) where we only need to invert the matrix $\mathbf{M}$.

The Chebyshev polynomials $T_n(x)$ are only valid in the interval $-1 \leq x \leq 1$, for this reason we rescale the equations (1) y (2) making the change $\tilde{x} = \frac{x}{L}$.

To solve the Schrödinger equation we use an ADI method (Alternating Direction Implicit), which allows us to rewrite the Schrödinger equation (2) in function of the following system equations

$$\begin{aligned} e^{-\frac{i\Delta t}{4}\frac{\partial^2}{\partial x^2}} S(x) &= e^{\frac{i\Delta t}{4}\frac{\partial^2}{\partial x^2}} \phi(t,x) \\ e^{\frac{i\Delta t}{2}V} \phi(t+\Delta t, x) &= e^{-\frac{i\Delta t}{2}V} S(x) \end{aligned} \tag{6}$$

We expanded the exponential to the lowest significant order in Δt, and discretized the second spatial partial derivative with second order centered Finite Difference method to obtain

$$\left(1 - \frac{i\Delta t}{4\Delta x}\delta_i^2\right) S_i = \left(1 + \frac{i\Delta t}{4\Delta x}\delta_i^2\right) \phi_i^n \tag{7}$$

$$\left(1 + \frac{i\Delta t}{2} V_i^{n+\frac{1}{2}}\right) \phi_i^{n+1} = \left(1 - \frac{i\Delta t}{2} V_i^{n+\frac{1}{2}}\right) S_i \tag{8}$$

with $\delta_i^2 = \delta_{x_i}^2$ and $\delta_{x_i}^2 f(x_i) = f(x - i + h) - 2f(x_i) + f(x_i - h)$. The subscript i labels the position $x_i = -1 + i\Delta x$ and the superscript n labels the time step $t_n = n\Delta t$, with $V_i^{n+1/2} = 3/2V_i^n - 1/2V_i^{n-1}$. First, we solve the equation (7)

$$\left(1 + \frac{i\Delta t}{2}H\right) S_i \quad = \quad \left(1 - \frac{i\Delta t}{2}H\right) \phi_i^n$$

with $H = -\frac{\partial^2}{\partial x^2}$, if we consider $S_i \sim \phi_i^{n+1}$ we can express it as follow

$$i\dot{\phi}_i^{n+1} = H\phi_i^n \tag{9}$$

we can solve equation (9) by using Method of Lines, and in this way we obtain the values of S_i . Then, from equation (8) we can easily calculate the values of ϕ_i^{n+1}, because we only need V and S.

SP system in the Fourier space

The Fourier representations for the potential $V(x)$ and $\phi(t,x)$ are

$$\phi(t,x) = \sum_{k=-N}^{N} \phi_k(t) e^{ikx}, \tag{10}$$

$$V(x) = \sum_{m=-N}^{N} V_k e^{imx}, \tag{11}$$

substituting these expressions in the SP system (1,2)

$$i \sum_{k=-N}^{N} \dot{\phi}_k e^{ikx} = \frac{1}{2} \sum_{k=-N}^{N} k^2 \phi_k e^{ikx} + \sum_{m,k=-N}^{N} V_m \phi_k e^{i(m+k)x} \tag{12}$$

$$-\sum_{m=-N}^{N} m^2 V_m e^{imx} = \sum_{k,p=-N}^{N} \phi_k^* \phi_p e^{i(p-k)x} \tag{13}$$

we multiply by e^{-inx} and integrate them onto x, to obtain

$$i\dot{\phi}_n = \frac{n^2}{2}\phi_n + \sum_{m+k=n} V_m \phi_k \tag{14}$$

$$V_n = -\frac{1}{n^2} \sum_{p-k=n} \phi_k^* \phi_p, \tag{15}$$

with $-N \leq n \leq N$ and $n \neq 0$. As we can see, equation (14) is a first order temporal partial derivative for ϕ which we solve using an integrator like a *Runge-Kutta*. The equation (15) is a recurrence expression for the values of the potential V, because it it enough to know the values of the function ϕ.

Equilibrium Configuration

We can solve the SP system to obtain the nonsingular self-gravitating configuration the so called equilibrium configurations which has the form

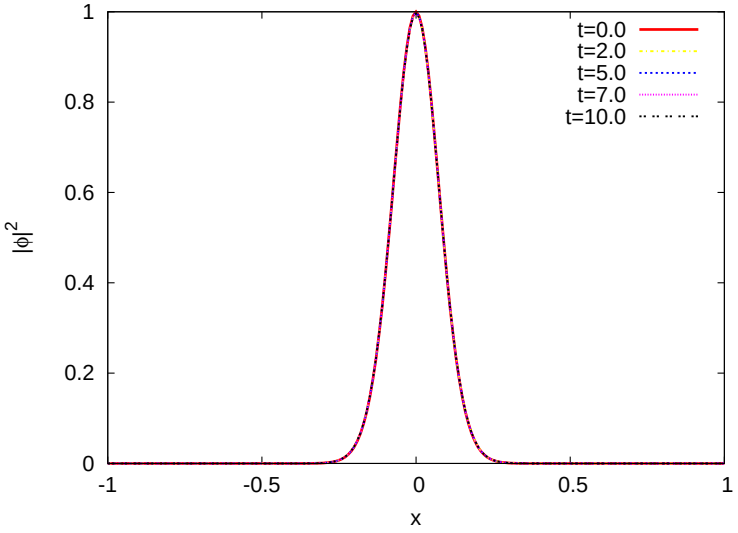
$$\phi(x,t) = \psi(x)e^{-i\omega t} \tag{16}$$

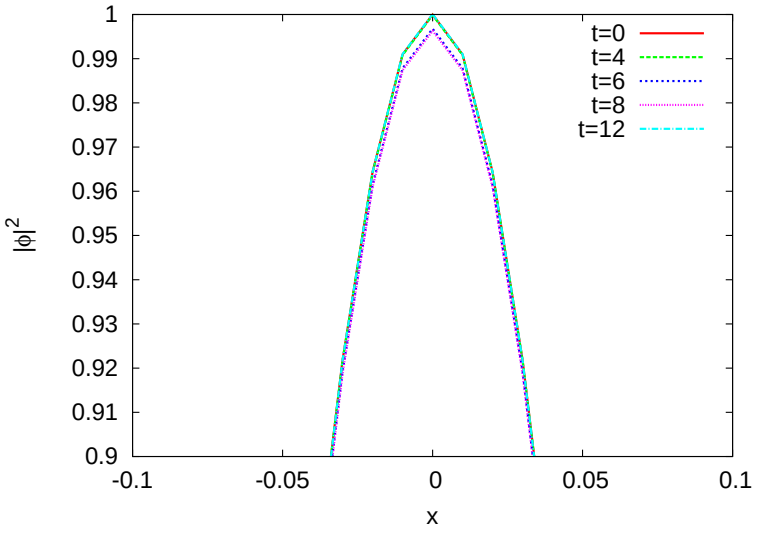
When we substitute the last expression into the SP system (1-2), we obtain an eigenvalue problem

$$\psi^{''} = 2\psi(x)U, \qquad U^{''} = \psi^2(x) \tag{17}$$

with $\psi(0) = 1$, $\psi^{'}(0) = 0$, $U(0) = U_0$ y $U^{'}(0) = 0$. To find the proper value of U_0 we use the Shooting Method.

Equilibrium Configuration





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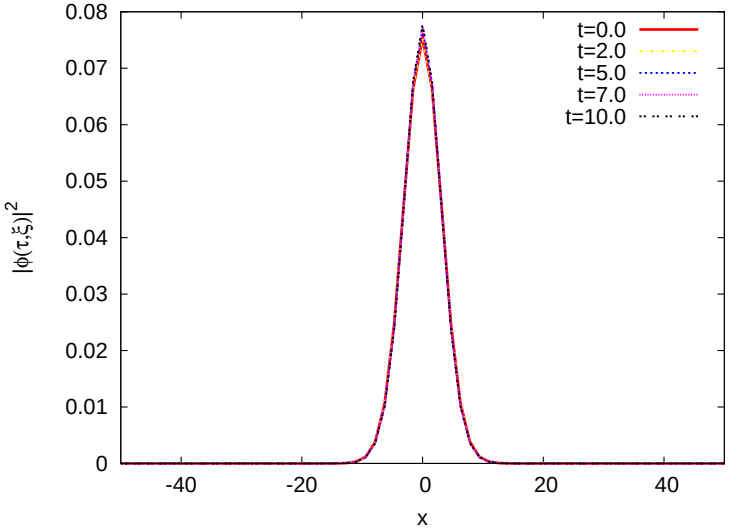

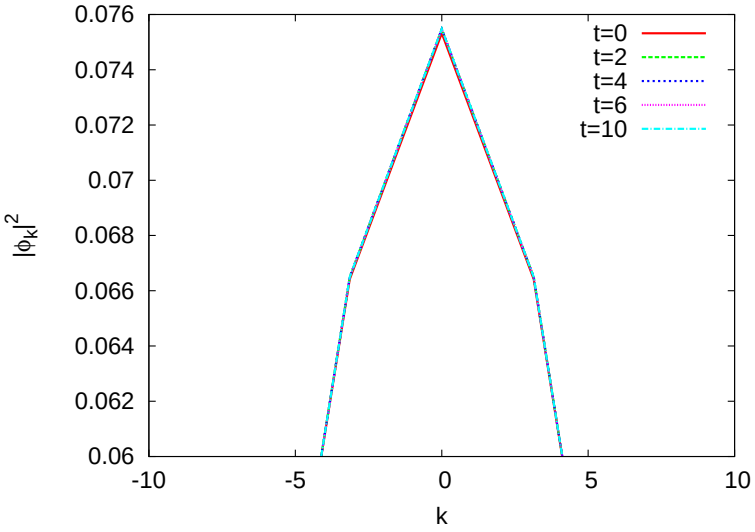

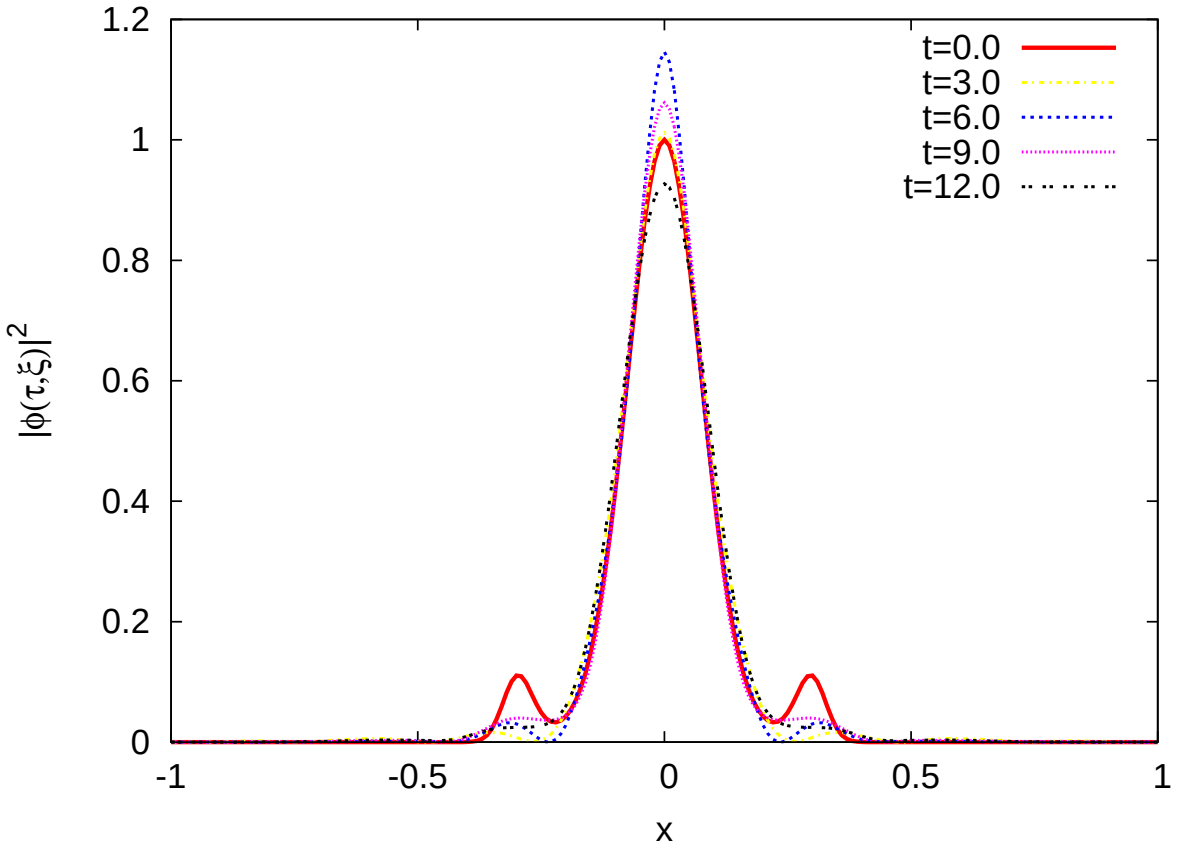
1.2
1
0.8
0.6
0.4
0.2
0
-1
-0.5
0
0.5
1
x
|ϕ(τ,ξ)|2
t=0.0
t=3.0
t=6.0
t=9.0
t=12.0

Final Comments

We constructed the equilibrium configurations of the SP system which were employed as initial conditions for our time evolution code. We observed that as these configurations evolve, their mass density is conserved in both, the physical and Fourier spaces. When an equilibrium configuration is perturbed, we expect it to eventually migrate to another equilibrium configuration.