Non-linear structure in the Universe
Cosmology on the Beach

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Limited options

- Beyond a certain scale, linear perturbation theory breaks down
  - Definition of “non-linear scale”?
- At this point we have few options:
  - Analytical models of non-linear growth.
    - Zel’dovich approximation.
    - Spherical top-hat collapse.
  - Perturbation theory.
    - Realm of validity? Convergence criterion?
    - Good for small corrections to almost linear problems.
  - Direct simulation.
    - Numerical convergence.
    - What models to run?
    - Missing physics.

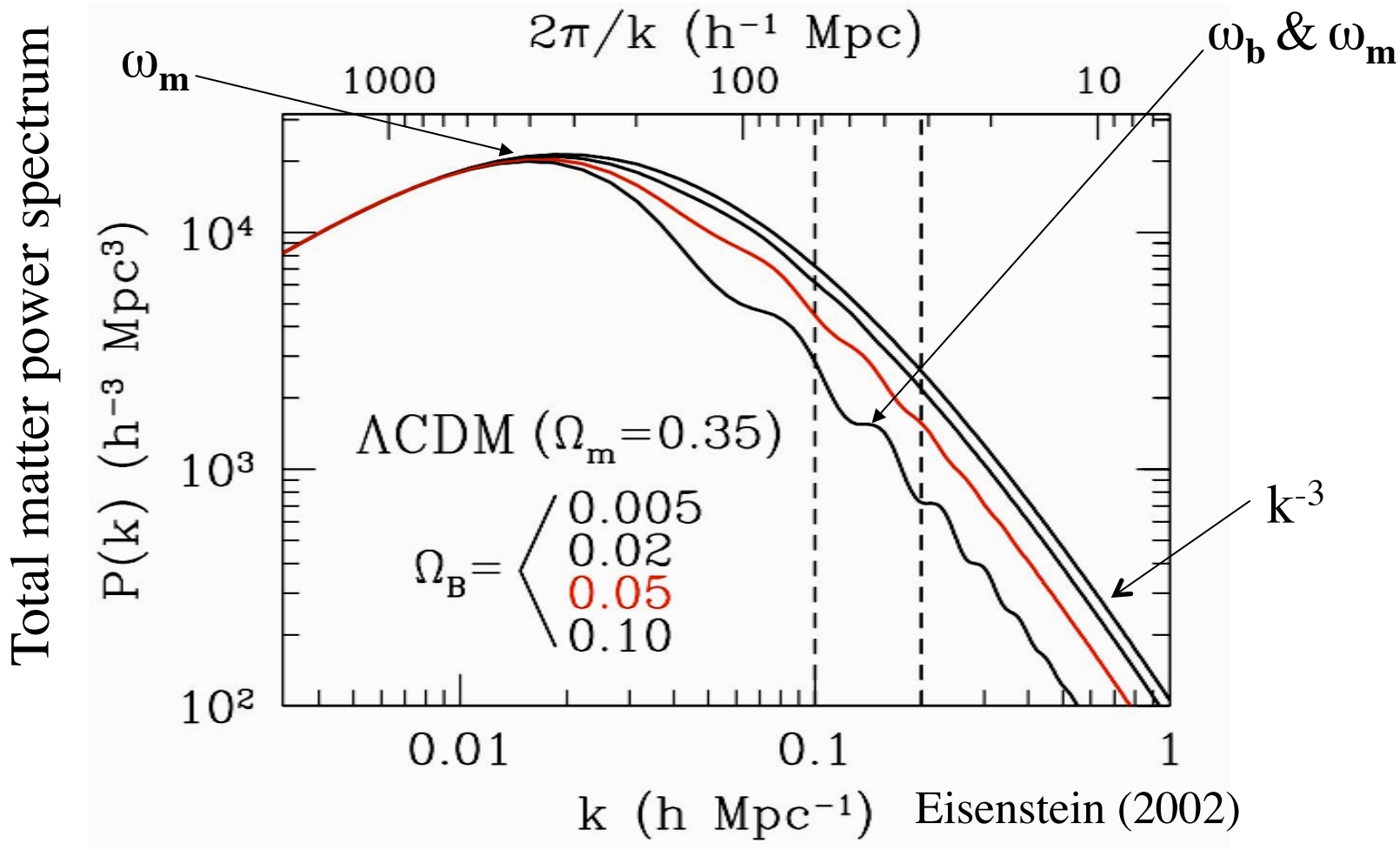
Notation

$$
\begin{aligned}
\delta(\mathbf{x}) &= \frac{\rho(\mathbf{x}) - \bar{\rho}}{\bar{\rho}} = \frac{\delta\rho}{\rho}(\mathbf{x}) \\
\delta(\mathbf{k}) &= \int d^3x \; \delta(\mathbf{x})\, e^{i\mathbf{k}\cdot\mathbf{x}} \\
\langle \delta(\mathbf{k})\delta^\star(\mathbf{k}') \rangle &= (2\pi)^3 \delta_D(\mathbf{k} - \mathbf{k}') P(k) \\
\Delta^2(k) &= \frac{k^3 P(k)}{2\pi^2} \\
\xi(x) &= \int \frac{d^3k}{(2\pi)^3} P(k) e^{i\mathbf{k}\cdot\mathbf{x}} \\
&= \int \frac{dk}{k} \; \Delta^2(k) j_0(kr)
\end{aligned}
$$

Linear PT

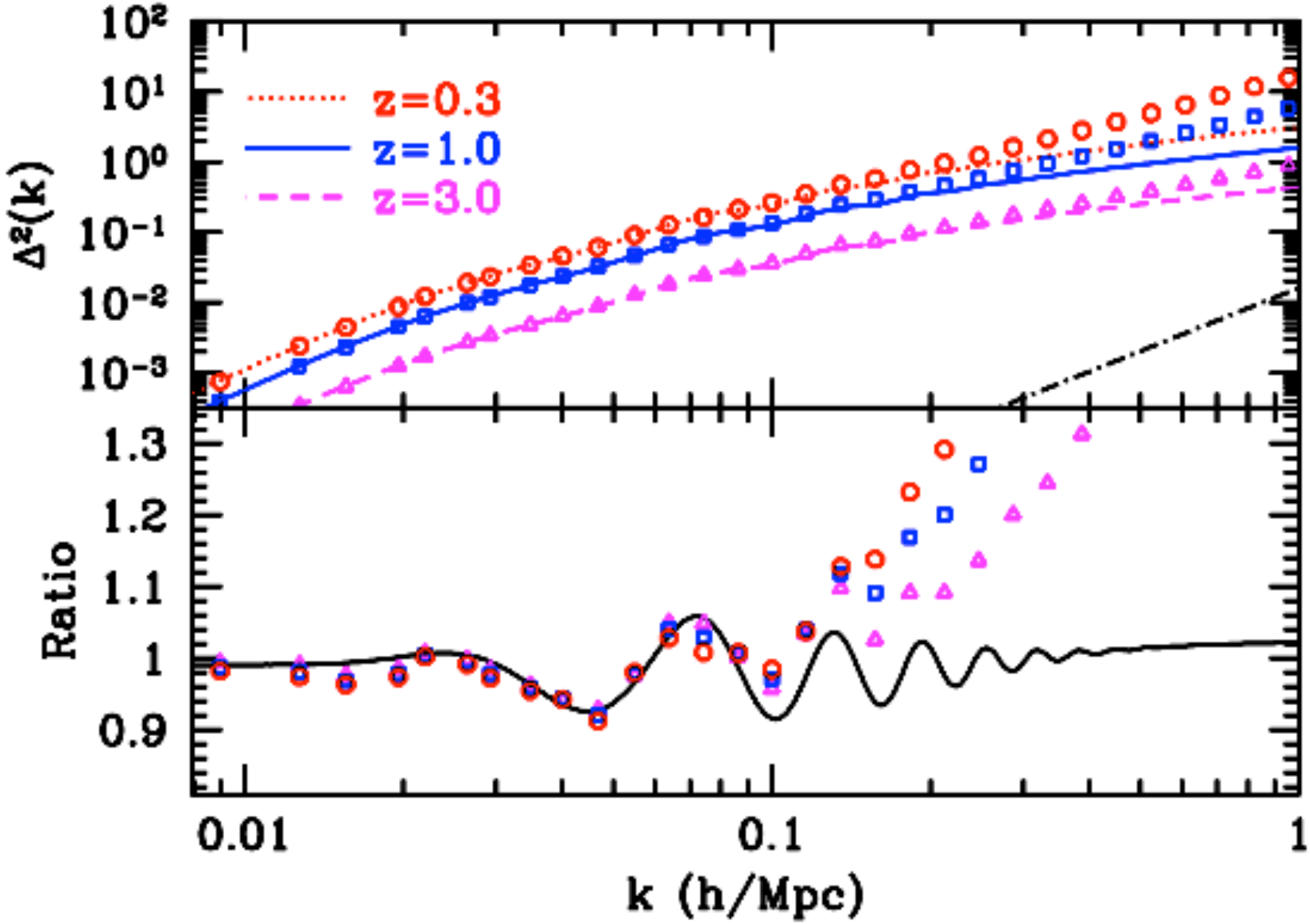
- For many scales and most of age of Universe linear perturbation theory is valid.
- Transfer function, *T(k)*, encodes 14Gyr of evolution.
  - $\delta_{today}(k) \sim (\text{growth}) \times T(k)\delta_{init}(k)$.
  - Main features RD->MD->ΛD.
  - Structure only grows when matter dominates energy density of Universe.

Matter power spectrum: $P_L(k)$



Eisenstein (2002)

Non-linearity



Scale of non-linearity

- There are several ways to define a "scale" of non-linearity.
- Where $\Delta^2(k)=1$ (or ½, or …).
  - Dangerous when $\Delta^2(k)$ is very flat.
- By the rms linear theory displacement.

$$R_{\rm nl}^2 \propto \frac{1}{k_{\rm nl}^2} \propto \int \frac{dk}{k} \frac{\Delta^2(k)}{k^2} \propto \int dk\ P(k)$$

- Where the 2nd order correction to some quantity is 1% (10%) of the 1st order term.

“Weak” non-linearity: Cosmological perturbation theory

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Perturbation theory

- There is no reason (in principle) to stop at linear order in perturbation theory.
  - Can expand to all orders: $\delta=\delta^{(1)}+\delta^{(2)}+\delta^{(3)}+...$
  - Can sum subsets of terms.
  - Usefulness/convergence of such an expansion not always clear.
- Consider **only** dark matter and **assume** we are in the single-stream limit.

Peebles (1980), Juszkiewicz (1981), Goroff++(1986), Makino++(1992), Jain&Bertschinger(1994), Fry (1994).
Reviews/comparison with N-body:
Bernardeau++(2002; Phys. Rep. 367, 1).
Carlson++(2009; PRD 80, 043531)

Equations of motion

Under these approximations, and assuming Ω_m=1

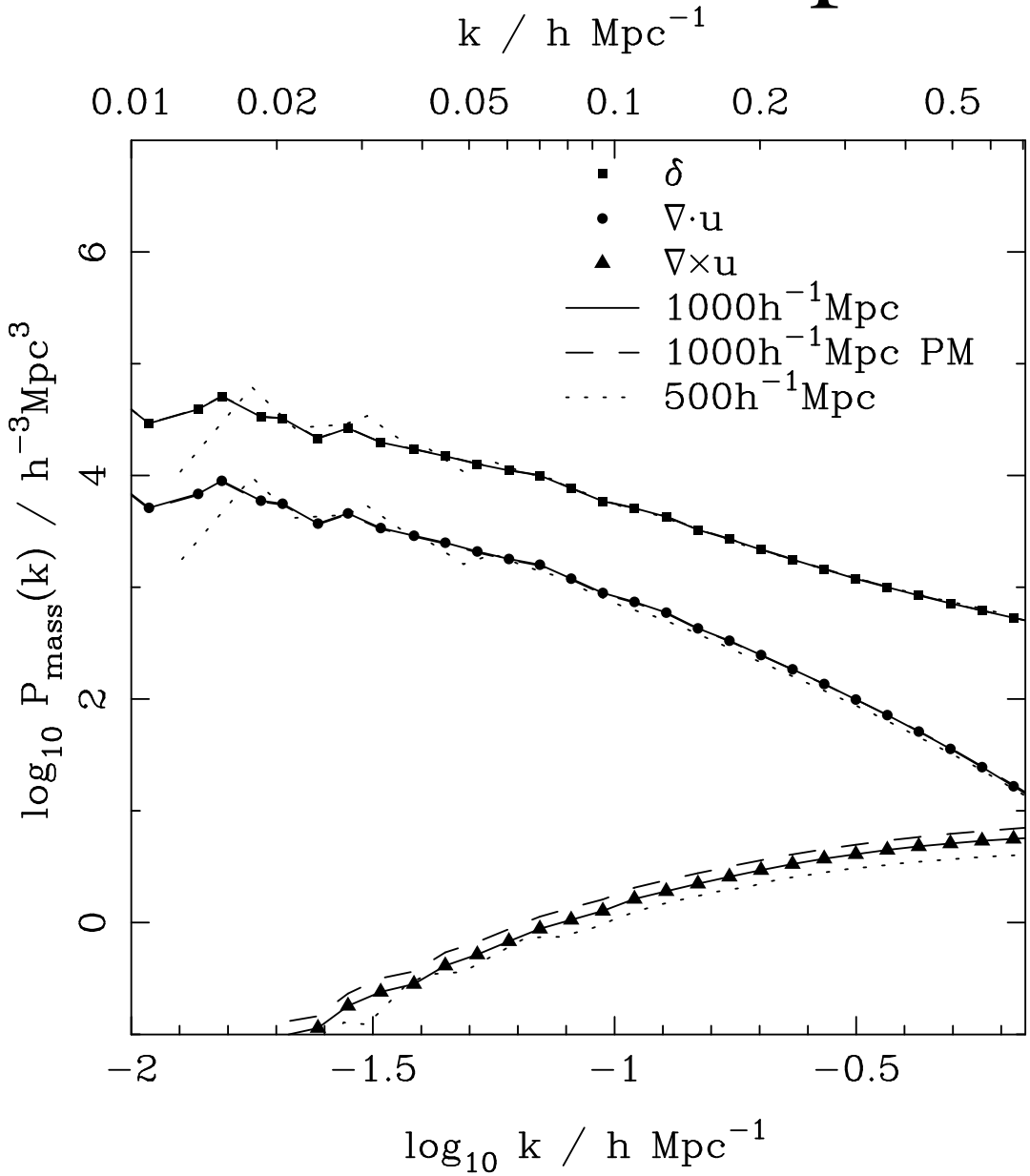
$$\begin{aligned} \frac{\partial \delta}{\partial \tau} + \vec{\nabla} \cdot [(1+\delta)\vec{v}] &= 0 \\ \frac{\partial \vec{v}}{\partial \tau} + \mathcal{H}\vec{v} + \left(\vec{v} \cdot \vec{\nabla}\right)\vec{v} &= -\vec{\nabla}\Phi \\ \nabla^2 \Phi &= \tfrac{3}{2}\mathcal{H}^2 \delta \end{aligned}$$

Gauge?

- Very familiar looking fluid equations
  - means we can borrow methods/ideas from other fields.
- Note the quadratic nature of the non-linearity.
- Since equations are now non-linear, can't use superposition of (exact) solutions even if they could be found!
- Proceed by perturbative expansion.

Velocities are ≈ potential flow

Percival & White (2009)



Assume that *v* comes from a potential flow (self-consistent; curl[v] ~a^{-1} at linear order) then it is totally specified by its divergence, θ.

Go into Fourier space

Putting the quadratic terms on the rhs and going into Fourier space:

Div

$\mathbf{v}\sim(\mathbf{q}/q^2)\theta$

Product=
Convolution

$$\frac{\partial\delta(\vec{k})}{\partial\tau}+\theta(\vec{k}) = -\int\frac{d^3q}{(2\pi)^3}\frac{\vec{k}\cdot\vec{q}}{q^2}\,\theta(\vec{q})\delta(\vec{k}-\vec{q}),$$

$$\frac{\partial\theta(\vec{k})}{\partial\tau}+\mathcal{H}\theta(\vec{k})+\frac{3}{2}\Omega_m\mathcal{H}^2\delta(\vec{k}) = -\int\frac{d^3q}{(2\pi)^3}\frac{k^2\,\vec{q}\cdot(\vec{k}-\vec{q})}{2q^2|\vec{k}-\vec{q}|^2} \times \theta(\vec{q})\theta(\vec{k}-\vec{q}).$$

Linear order

- To lowest order in δ and θ:

$$\delta_L(\mathbf{k}, z) = \frac{D(z)}{D(z_i)}\,\delta_i(\mathbf{k})$$

$$\theta_L(\mathbf{k}, z) = -f(z)\mathcal{H}(z)\,\frac{D(z)}{D(z_i)}\,\delta_i(\mathbf{k})$$

- with $f(z) \sim \Omega_m^{0.6} = 1$ for $\Omega_m = 1$ and $D(a) \sim a$.
- Decaying mode, $\delta \sim a^{-3/2}$, has to be zero for δ to be well-behaved as a->0.
- Define $\delta_0 = \delta_L(k, z=0)$.

Standard perturbation theory

- Develop δ and θ as power series:

$$\delta(\mathbf{k}) = \sum_{n=1}^{\infty} a^n \delta^{(n)}(\mathbf{k})$$

$$\theta(\mathbf{k}) = -\mathcal{H} \sum_{n=1}^{\infty} a^n \theta^{(n)}(\mathbf{k})$$

- then the $\delta^{(n)}$ can be written

$$\delta^{(n)}(\mathbf{k}) = \int \frac{d^3q_1 d^3q_2 \cdots d^3q_n}{(2\pi)^{3n}} (2\pi)^3 \delta_D \left(\sum \mathbf{q}_i - \mathbf{k} \right) \times F_n \left(\{\mathbf{q}_i\} \right) \delta_0(\mathbf{q}_1) \cdots \delta_0(\mathbf{q}_n)$$

- with a similar expression for $\theta^{(n)}$.
- The F_n and G_n are just ratios of dot products of the ***q***s and obey simple recurrence relations.

Recurrence relations I

- Plugging the expansion into our equations and using
  - $(d/d\tau)a^n = nHa^n$
  - $(d/d\tau)H = (-1/2)H^2$ for EdS
- we have (canceling H from both sides):

$$n\delta^{(n)} + \theta^{(n)} = -\int \frac{d^3q_1}{(2\pi)^3}\frac{d^3q_2}{(2\pi)^3}(2\pi)^3\delta(\vec{k}-\vec{q}_1-\vec{q}_2)\frac{\vec{k}\cdot\vec{q}_1}{q_1^2}\sum_{m=1}^{n-1}\theta_m(\vec{q}_1)\delta_{n-m}(\vec{q}_2)$$

$$3\delta^{(n)} + (2n+1)\theta^{(n)} = -\int \frac{d^3q_1}{(2\pi)^3}\frac{d^3q_2}{(2\pi)^3}(2\pi)^3\delta(\vec{k}-\vec{q}_1-\vec{q}_2)\frac{k^2(\vec{q}_1\cdot\vec{q}_2)}{q_1^2q_2^2}\sum_{m=1}^{n-1}\theta_m(\vec{q}_1)\theta_{n-m}(\vec{q}_2)$$

Recurrence relations II

- Which we can rewrite

$$\delta^{(n)} = \frac{(2n+1)A_n - B_n}{(2n+3)(n-1)} \quad , \quad \theta^{(n)} = \frac{-3A_n + nB_n}{(2n+3)(n-1)}$$

- where A_n and B_n are the rhs mode-coupling integrals.
- This generates recursion relations for the F_n and G_n (because of the sums in A_n and B_n)

$$F_n = \sum_{m=1}^{n-1} \frac{G_m}{(2n+3)(n-1)} \left[(2n+1)\frac{\vec{k}\cdot\vec{k}_1}{k_1^2} F_{n-m} + \frac{k^2(\vec{k}_1\cdot\vec{k}_2)}{k_1^2 k_2^2} G_{n-m} \right]$$

$$G_n = \sum_{m=1}^{n-1} \frac{G_m}{(2n+3)(n-1)} \left[3\frac{\vec{k}\cdot\vec{k}_1}{k_1^2} F_{n-m} + n\frac{k^2(\vec{k}_1\cdot\vec{k}_2)}{k_1^2 k_2^2} G_{n-m} \right]$$

Example: 2nd order

- The coupling function:

$$F_2(\mathbf{k}_1, \mathbf{k}_2) = \frac{5}{7} + \frac{2}{7}\frac{(\mathbf{k}_1 \cdot \mathbf{k}_2)^2}{k_1^2 k_2^2} + \frac{(\mathbf{k}_1 \cdot \mathbf{k}_2)}{2}\left(k_1^{-2} + k_2^{-2}\right)$$

- where we have symmetrized the function in terms of its arguments.
  - Note: this function peaks when $k_1 \sim k_2 \sim k/2$.
  - This will be important later.

Formal development

- We can make the expressions above more formal by defining η=ln(a) and

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = e^{-\eta} \begin{pmatrix} \delta \\ -\theta/\mathcal{H} \end{pmatrix}$$

- then writing

$$\partial_\eta \phi_a = -\Omega_{ab}\phi_b + e^\eta \gamma_{abc}\phi_b\phi_c$$

- with the obvious definitions of Ω and γ.
- We can also define P~<$\phi\phi$>, B~<$\phi\phi\phi$> so e.g.

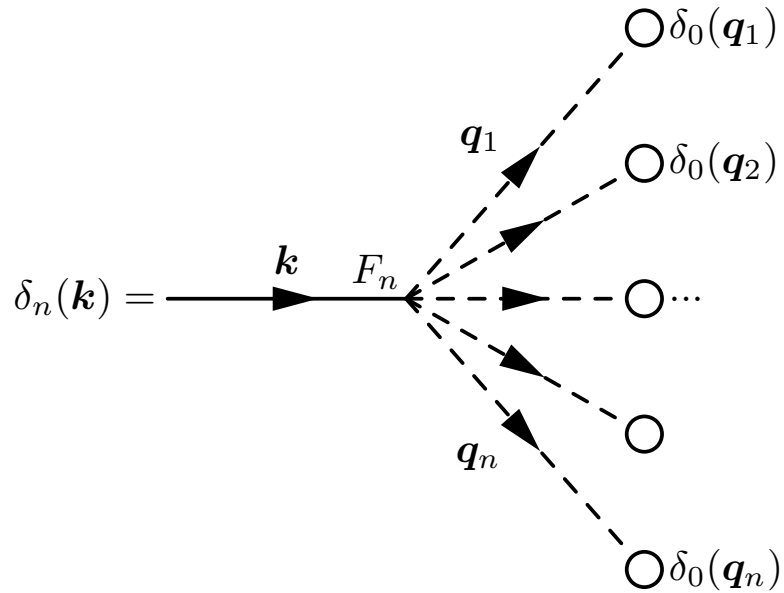
$$\partial_\eta P_{ab} = -\Omega_{ac}P_{cb} - \Omega_{bc}P_{ac} + e^\eta \int d^3q \left[\gamma_{acd}B_{bcd} + B_{acd}\gamma_{bcd}\right]$$

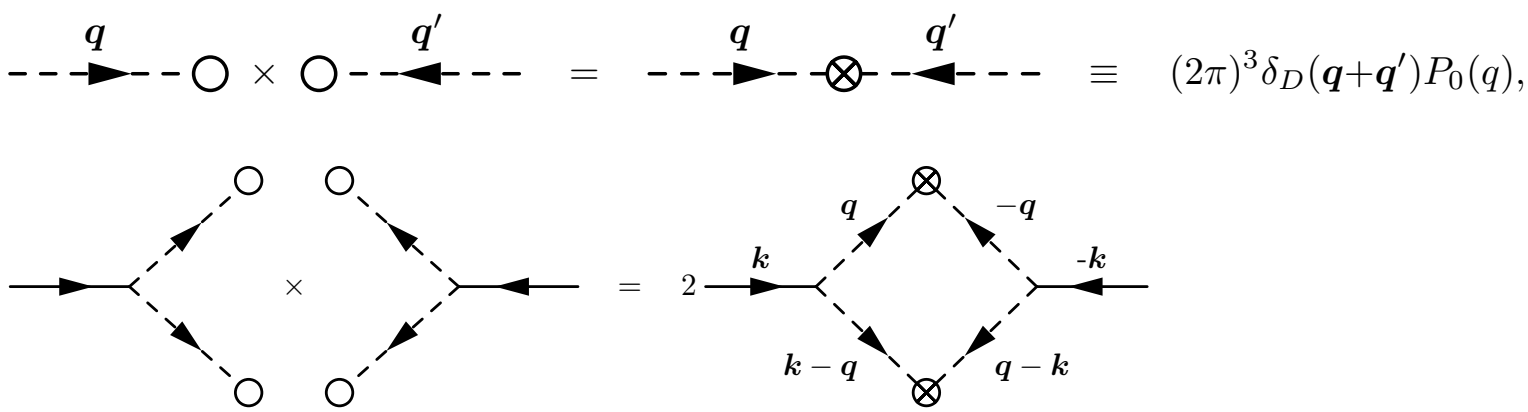
Power spectrum

- If the initial fluctuations are Gaussian only expectation values even in δ_0 survive:
  - $P(k) \sim \langle[\delta^{(1)}+\delta^{(2)}+\delta^{(3)}+\ldots][\delta^{(1)}+\delta^{(2)}+\delta^{(3)}+\ldots]\rangle$
  - $= P^{(1,1)} + 2P^{(1,3)} + P^{(2,2)} + \ldots$
- with terms like $\langle\delta^{(1)}\delta^{(2)}\rangle$ vanishing because they reduce to $\langle\delta_0\delta_0\delta_0\rangle$.

Perturbation theory: diagrams

Just as there is a diagrammatic short-hand for perturbation theory in quantum field theory, so there is in cosmology:





$$= 2\int \frac{d^3q}{(2\pi)^3} F_2(\boldsymbol{q}, \boldsymbol{k}-\boldsymbol{q}) F_2(-\boldsymbol{q}, \boldsymbol{q}-\boldsymbol{k}) P_0(q) P_0(|\boldsymbol{k}-\boldsymbol{q}|)$$

Example: 2nd order

$$P^{(1,3)}(k) = \frac{1}{252}\frac{k^3}{4\pi^2}P_L(k)\int_0^\infty dr\ P_L(kr)\left[\frac{12}{r^2} - 158 + 100r^2 - 42r^4 + \frac{3}{r^2}(r^2-1)^3(7r^2+2)\ln\left|\frac{1+r}{1-r}\right|\right],$$

$$P^{(2,2)}(k) = \frac{1}{98}\frac{k^3}{4\pi^2}\int_0^\infty dr\ P_L(kr)\int_{-1}^{1} dx\ P_L\left(k\sqrt{1+r^2-2rx}\right) \times \frac{(3r+7x-10rx^2)^2}{(1+r^2-2rx)^2}.$$

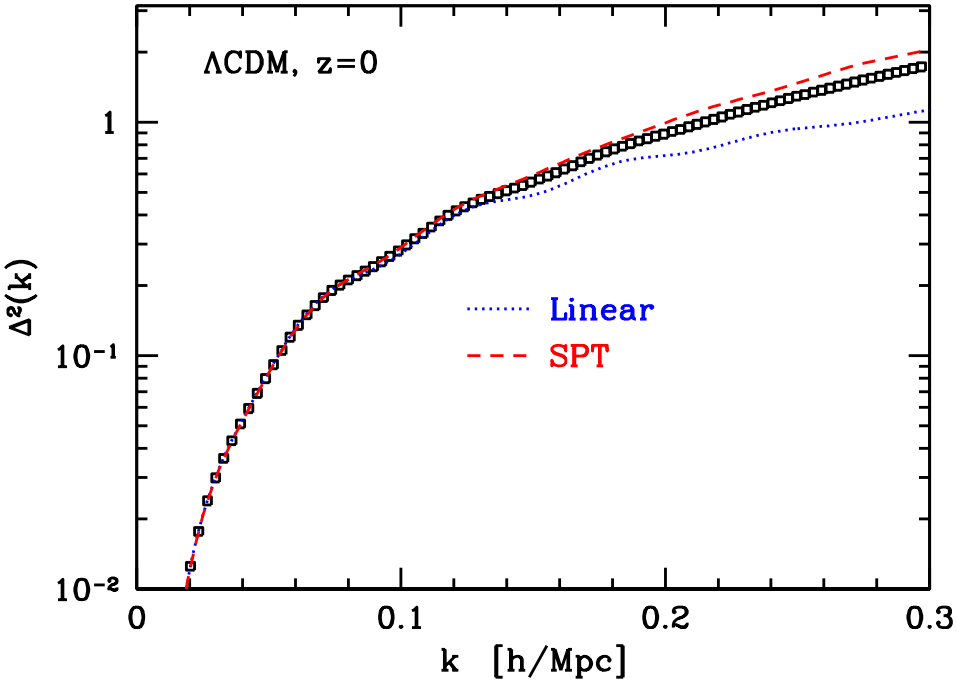
Perturbation theory enables the generation of truly impressive looking equations which arise from simple angle integrals. Like Feynman integrals, they are simple but look erudite!

Example: 2nd order

- At low *k*, $P^{(2,2)}$ is positive and $P^{(1,3)}$ is negative
  - Large cancellation.
- For large *k* total contribution is negative:
  - $P^{(2,2)} \sim (1/4)\, k^2 \Sigma^2\, P_L(k)$
  - $P^{(1,3)} \sim -(1/2)\, k^2 \Sigma^2\, P_L(k)$
- Here Σ is the rms displacement (in each component) in linear theory.
  - It will come up again!!

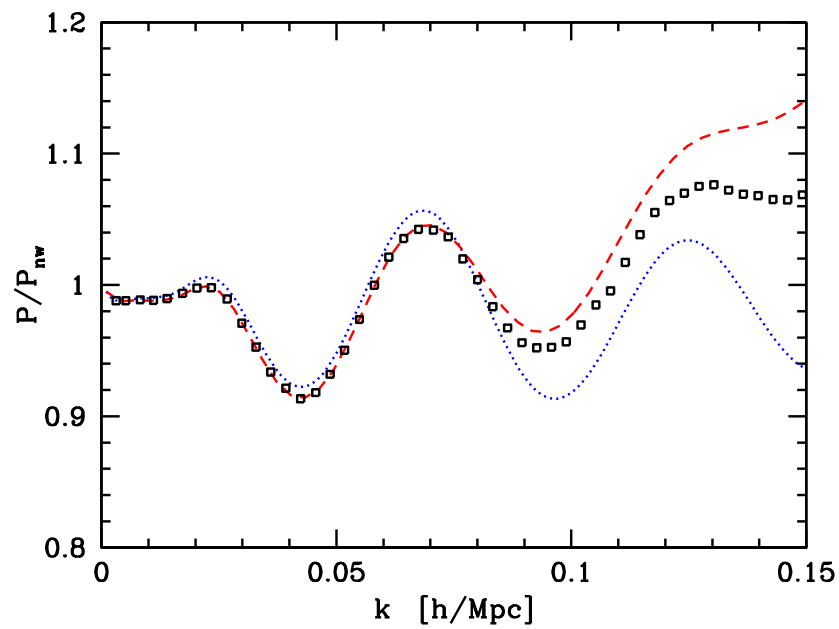
$$\Sigma^2 = \frac{1}{3\pi^2} \int_0^\infty dq\ P_L(q)$$

Example



The lowest order correction to the matter power spectrum at z=0 (1-loop SPT).

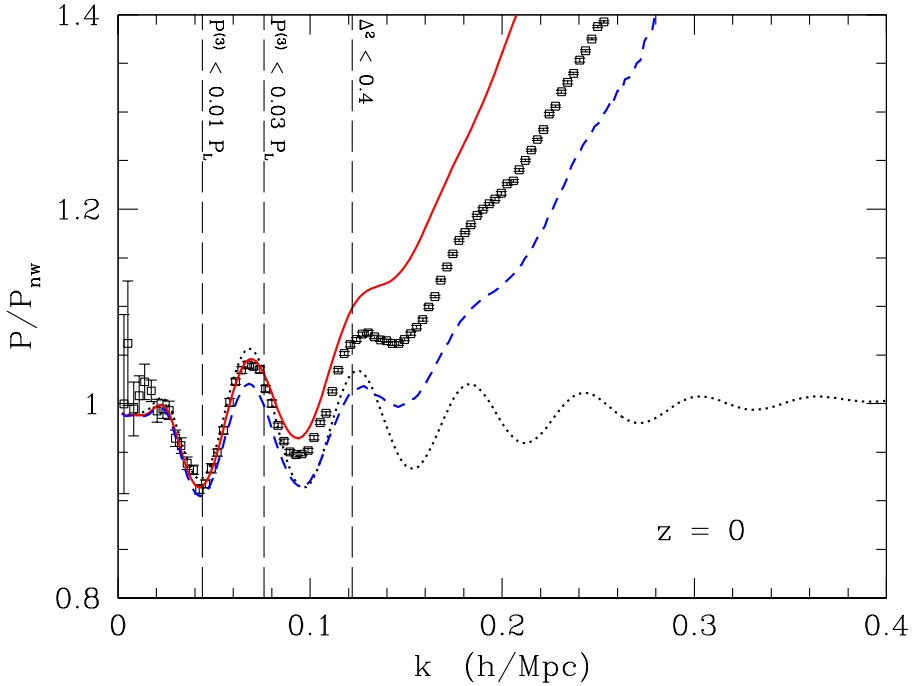
Note the improvement at low k where non-linear growth causes a <u>suppression</u> of power (pre-virialization).



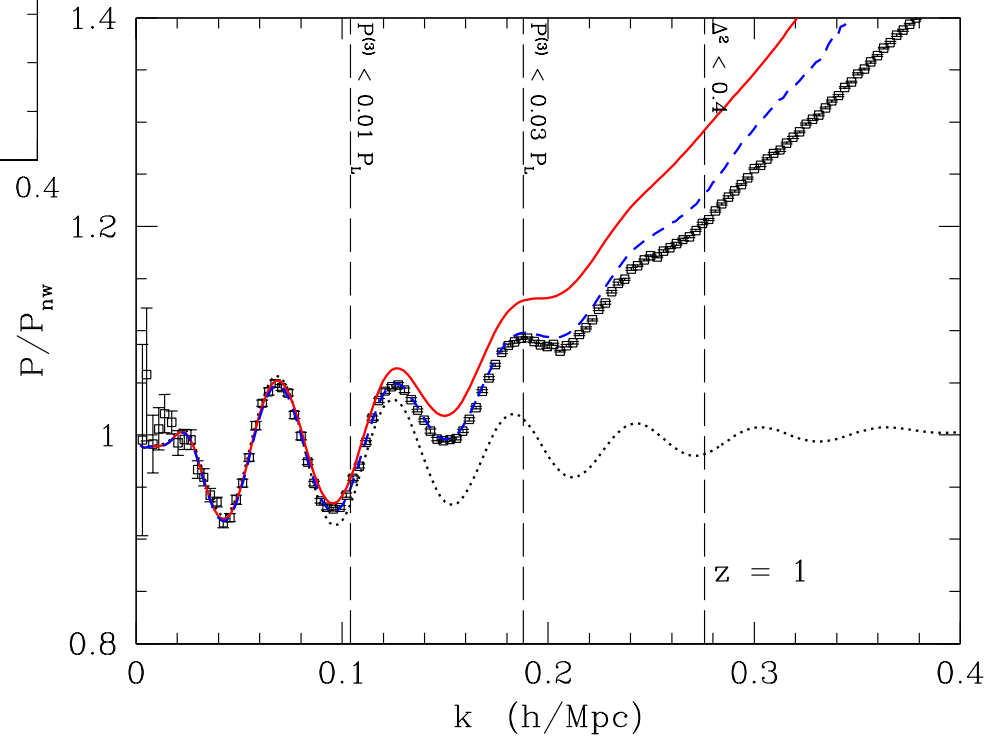
Beyond 2nd order

- Expressions for higher orders are easy to derive, especially using computer algebra packages.
- Using rotation symmetry the Nth order contribution requires mode coupling integrals of dimension 3N-1.
  - Best done using Monte-Carlo integration.
  - Prohibitive for very high orders.
  - Not clear this expansion is converging!

Comparison with exact results



Broad-band shape of P_L has been divided out to focus on more subtle features.



Linear

1st order correction

2nd order correction

Carlson++09

Including bias

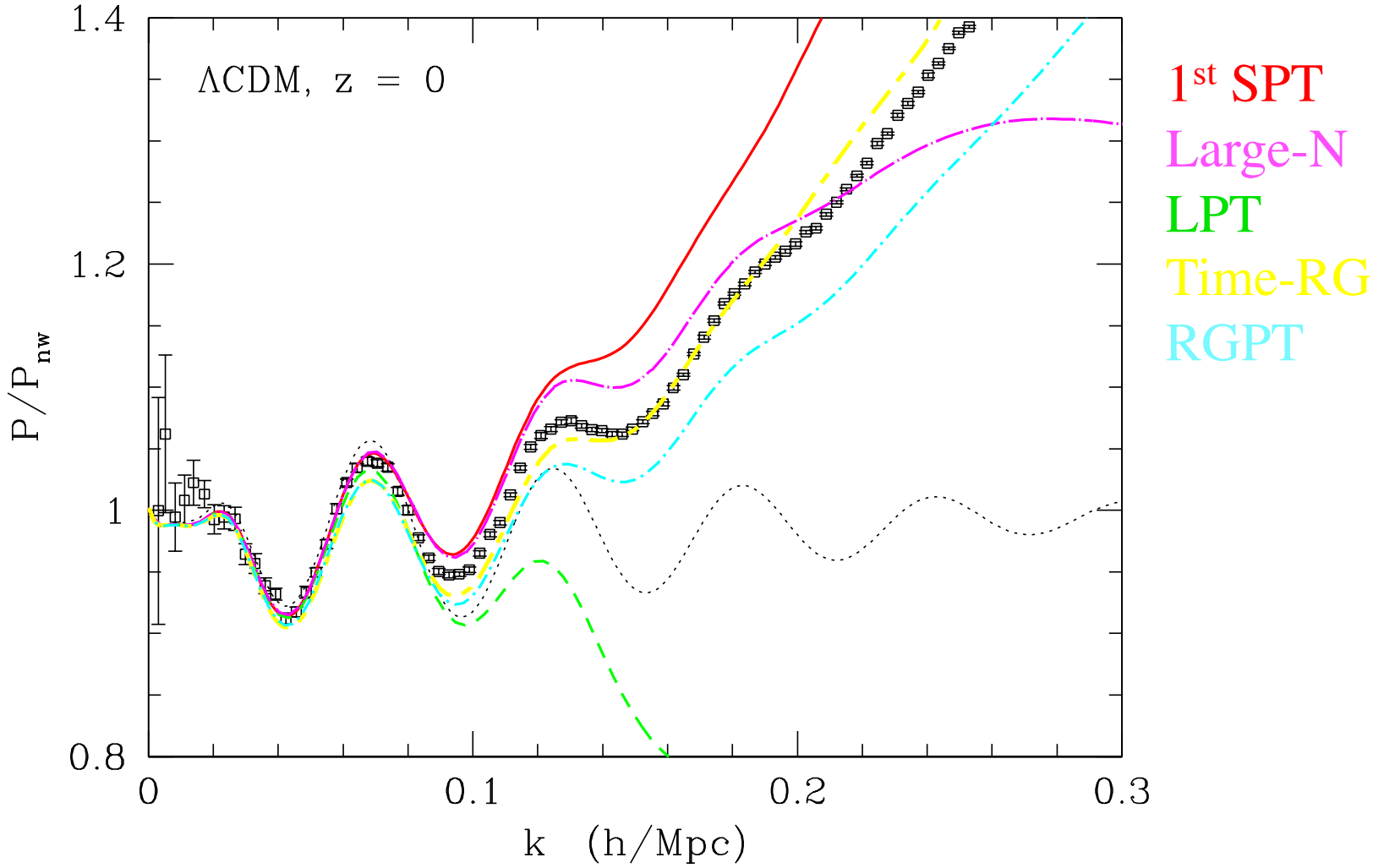
- Perturbation theory clearly cannot describe the formation of collapsed, bound objects such as dark matter halos.
- We can extend the usual thinking about "linear bias" to a power-series in the Eulerian density field:
  - $\delta_{obj} = \Sigma\, b_n(\delta^n/n!)$
- The expressions for P(k) now involve b_1 to lowest order, b_1 and b_2 to next order, etc.
  - The physical meaning of these terms is actually hard to figure out, and the validity of the defining expression is dubious, but this is the standard way to include bias in Eulerian perturbation theory.

Other methods

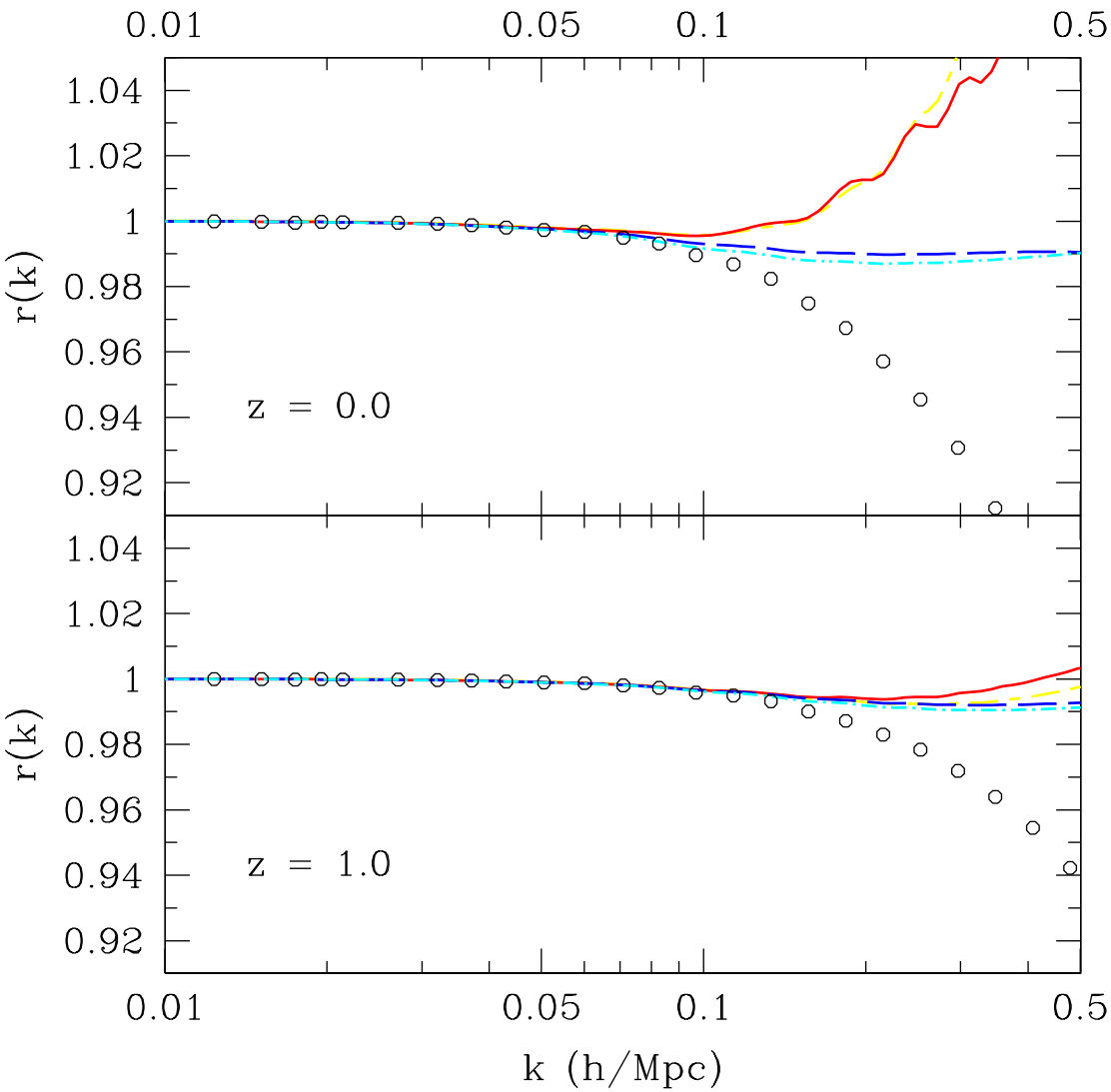
- Renormalized perturbation theory
  - A variant of “Dyson-Wyld” resummation.
  - An expansion in “order of complexity”.
- Closure theory
  - Write expressions for $(d/d\tau)P$ in terms of P, B, T, …
  - Approximate B by leading-order expression in SPT.
- Time-RG theory (& RGPT)
  - As above, but assume B=0
  - Good for models with $m_\nu>0$ where linear growth is scale-dependent.
- Path integral formalism
  - Perturbative evaluation of path integral gives SPT.
  - Large N expansion, 2PI effective action, steepest descent.
- Lagrangian perturbation theory

(see Carlson++09 for references)

Some other theories



Other statistics



PT makes predictions for other statistics as well. For example, the power spectra of the velocity and the density-velocity cross spectrum. Here it seems to do less well.

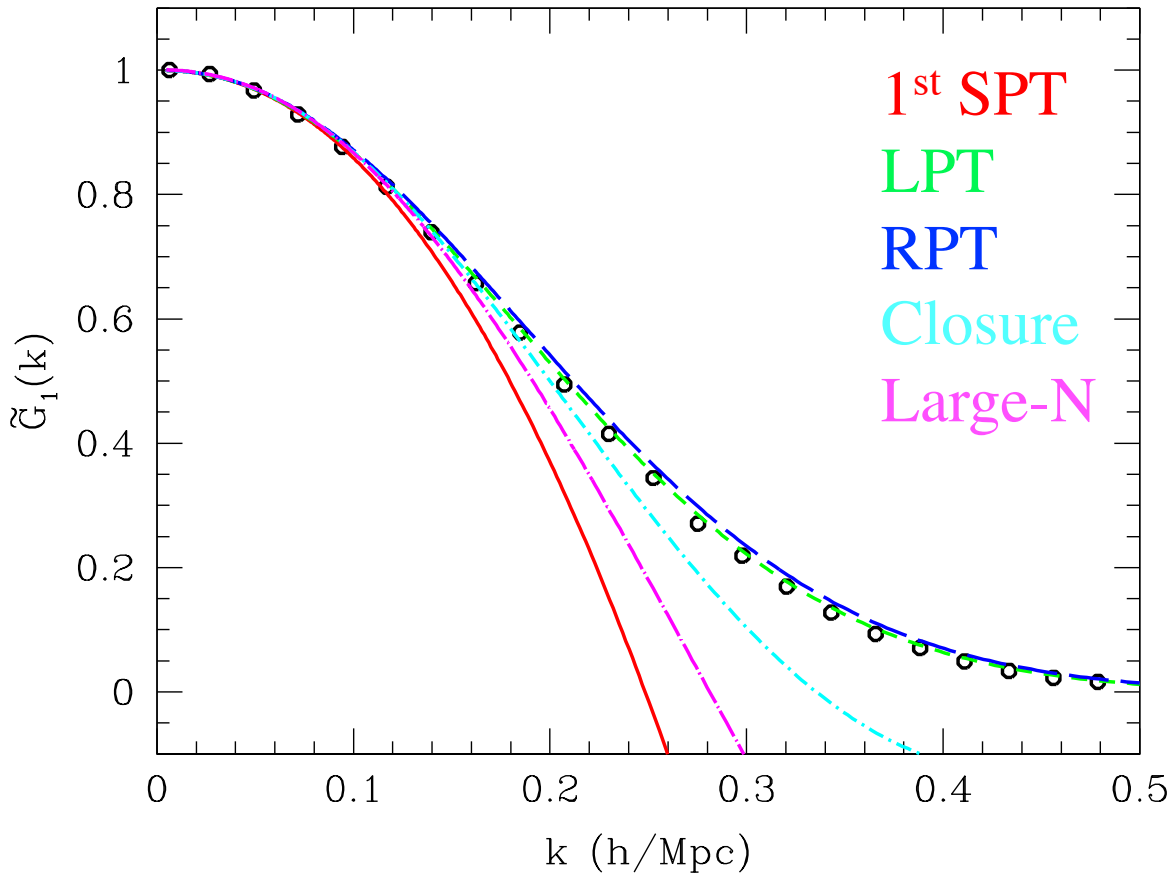
SPT

RPT

Closure

Time-RG

Some other quantities



The propagator, or

$$G(k) \propto \frac{\langle \delta_{NL} \delta_L^* \rangle}{\langle \delta_L \delta_L^* \rangle}$$

which measures the decoherence of the final density field due to non-linear evolution.

Carlson++09

Lagrangian perturbation theory

- A different approach to PT, which has been radically developed recently by Matsubara and is *very* useful for BAO.
  - Buchert89, Moutarde++91, Bouchet++92, Catelan95, Hivon++95.
  - Matsubara (2008a; PRD, 77, 063530)
  - Matsubara (2008b; PRD, 78, 083519)
- Relates the current (Eulerian) position of a mass element, **x**, to its initial (Lagrangian) position, **q**, through a displacement vector field, Ψ.

Lagrangian perturbation theory

$$\delta(\mathbf{x}) = \int d^3q \; \delta_D(\mathbf{x} - \mathbf{q} - \mathbf{\Psi}) - 1$$

$$\delta(\mathbf{k}) = \int d^3q \; e^{-i\mathbf{k}\cdot\mathbf{q}} \left(e^{-i\mathbf{k}\cdot\mathbf{\Psi}(\mathbf{q})} - 1 \right) \; .$$

$$\frac{d^2\mathbf{\Psi}}{dt^2} + 2H\frac{d\mathbf{\Psi}}{dt} = -\nabla_x \phi \left[\mathbf{q} + \mathbf{\Psi}(\mathbf{q})\right]$$

$$\begin{aligned} \mathbf{\Psi}^{(n)}(\mathbf{k}) &= \frac{i}{n!} \int \prod_{i=1}^{n} \left[\frac{d^3k_i}{(2\pi)^3} \right] \; (2\pi)^3 \delta_D \left(\sum_i \mathbf{k}_i - \mathbf{k} \right) \\ &\times \quad \mathbf{L}^{(n)}(\mathbf{k}_1, \cdots, \mathbf{k}_n, \mathbf{k}) \delta_0(\mathbf{k}_1) \cdots \delta_0(\mathbf{k}_n) \end{aligned}$$

Kernels

$$\mathbf{L}^{(1)}(\mathbf{p}_1) = \frac{\mathbf{k}}{k^2} \tag{1}$$

$$\mathbf{L}^{(2)}(\mathbf{p}_1, \mathbf{p}_2) = \frac{3}{7}\frac{\mathbf{k}}{k^2}\left[1 - \left(\frac{\mathbf{p}_1 \cdot \mathbf{p}_2}{p_1 p_2}\right)^2\right] \tag{2}$$

$$\mathbf{L}^{(3)}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) = \cdots \tag{3}$$

$$\mathbf{k} \equiv \mathbf{p}_1 + \cdots + \mathbf{p}_n$$

Standard LPT

- If we expand the exponential and keep terms consistently in δ_0 we regain a series $\delta=\delta^{(1)}+\delta^{(2)}+\ldots$ where $\delta^{(1)}$ is linear theory and e.g.

$$\begin{aligned}\delta^{(2)}(\mathbf{k}) &= \frac{1}{2}\int \frac{d^3k_1 d^3k_2}{(2\pi)^3}\delta_D(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})\delta_0(\mathbf{k}_1)\delta_0(\mathbf{k}_2) \\ &\times \left[\mathbf{k}\cdot\mathbf{L}^{(2)}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}) + \mathbf{k}\cdot\mathbf{L}^{(1)}(\mathbf{k}_1)\mathbf{k}\cdot\mathbf{L}^{(1)}(\mathbf{k}_2)\right]\end{aligned}$$

- which regains "SPT".
  - The quantity in square brackets is F_2.

$$F_2(\mathbf{k}_1,\mathbf{k}_2) = \frac{5}{7} + \frac{2}{7}\frac{(\mathbf{k}_1\cdot\mathbf{k}_2)^2}{k_1^2 k_2^2} + \frac{(\mathbf{k}_1\cdot\mathbf{k}_2)}{2}\left(k_1^{-2}+k_2^{-2}\right)$$

LPT power spectrum

- Alternatively we can use the expression for $\delta_{\mathbf{k}}$ to write

$$P(k) = \int d^3q \; e^{-i\vec{k}\cdot\vec{q}} \left(\left\langle e^{-i\vec{k}\cdot\Delta\vec{\Psi}} \right\rangle - 1 \right)$$

- where $\Delta\Psi=\Psi(\mathbf{q})-\Psi(0)$.
- Expanding the exponential and plugging in for $\Psi^{(n)}$ gives the usual results.
- **BUT** Matsubara suggested a different and very clever approach.

Cumulants

- The cumulant expansion theorem allows us to write the expectation value of the exponential in terms of the exponential of expectation values.
- Expand the terms $(\mathbf{k}\Delta\Psi)^N$ using the binomial theorem.
- There are two types of terms:
  - Those depending on Ψ at same point.
    - This is independent of position and can be factored out of the integral.
  - Those depending on Ψ at different points.
    - These can be expanded as in the usual treatment.

Example

- Imagine Ψ is Gaussian with mean zero.
- For such a Gaussian: $\langle e^{\Psi} \rangle = \exp[\sigma^2/2]$.

$$P(k) = \int d^3q e^{-i\mathbf{k}\cdot\mathbf{q}} \left(\left\langle e^{-ik_i \Delta\Psi_i(\mathbf{q})} \right\rangle - 1 \right)$$

$$\left\langle e^{-i\mathbf{k}\cdot\Delta\mathbf{\Psi}(q)} \right\rangle = \exp\left[-\frac{1}{2} k_i k_j \left\langle \Delta\Psi_i(\mathbf{q}) \Delta\Psi_j(\mathbf{q}) \right\rangle \right]$$

$$k_i k_j \left\langle \Delta\Psi_i(\mathbf{q}) \Delta\Psi_j(\mathbf{q}) \right\rangle = 2k_i^2 \langle \Psi_i^2(\mathbf{0}) \rangle - 2k_i k_j \xi_{ij}(\mathbf{q})$$

Keep exponentiated. Expand

Resummed LPT

- The first corrections to the power spectrum are then:

$$P(k) = e^{-(k\Sigma)^2/2}\left[P_L(k) + P^{(2,2)}(k) + \widetilde{P}^{(1,3)}(k)\right],$$

- where $P^{(2,2)}$ is as in SPT but part of $P^{(1,3)}$ has been "resummed" into the exponential prefactor.

- The exponential prefactor is identical to that obtained from
  - The peak-background split (Eisenstein++07)
  - Renormalized Perturbation Theory (Crocce++08).

Beyond real-space mass

- One of the more impressive features of Matsubara's approach is that it can gracefully handle both biased tracers and redshift space distortions.
- In redshift space, in the plane-parallel limit,

$$\mathbf{\Psi} \to \mathbf{\Psi} + \frac{\widehat{\mathbf{z}} \cdot \dot{\mathbf{\Psi}}}{H}\,\widehat{z} = R\,\mathbf{\Psi}$$

- In PT $\mathbf{\Psi}^{(n)} \propto D^n \Rightarrow R_{ij}^{(n)} = \delta_{ij} + nf\,\widehat{z}_i\widehat{z}_j$

- Again we're going to leave the zero-lag piece exponentiated so that the prefactor contains

$$k_i k_j R_{ia} R_{jb} \delta_{ab} = (k_a + fk\mu\widehat{z}_a)\,(k_a + fk\mu\widehat{z}_a) = k^2\left[1 + f(f+2)\mu^2\right]$$

- while the $\xi(r)$ piece, when FTed, becomes the usual Kaiser expression plus higher order terms.

Beyond real-space mass

- One of the more impressive features of Matsubara's approach is that it can gracefully handle both biased tracers and redshift space distortions.

- For bias local in Lagrangian space:

$$\delta_{\rm obj}(\mathbf{x}) = \int d^3q \; F\left[\delta_L(\mathbf{q})\right] \delta_D(\mathbf{x} - \mathbf{q} - \mathbf{\Psi})$$

- we obtain

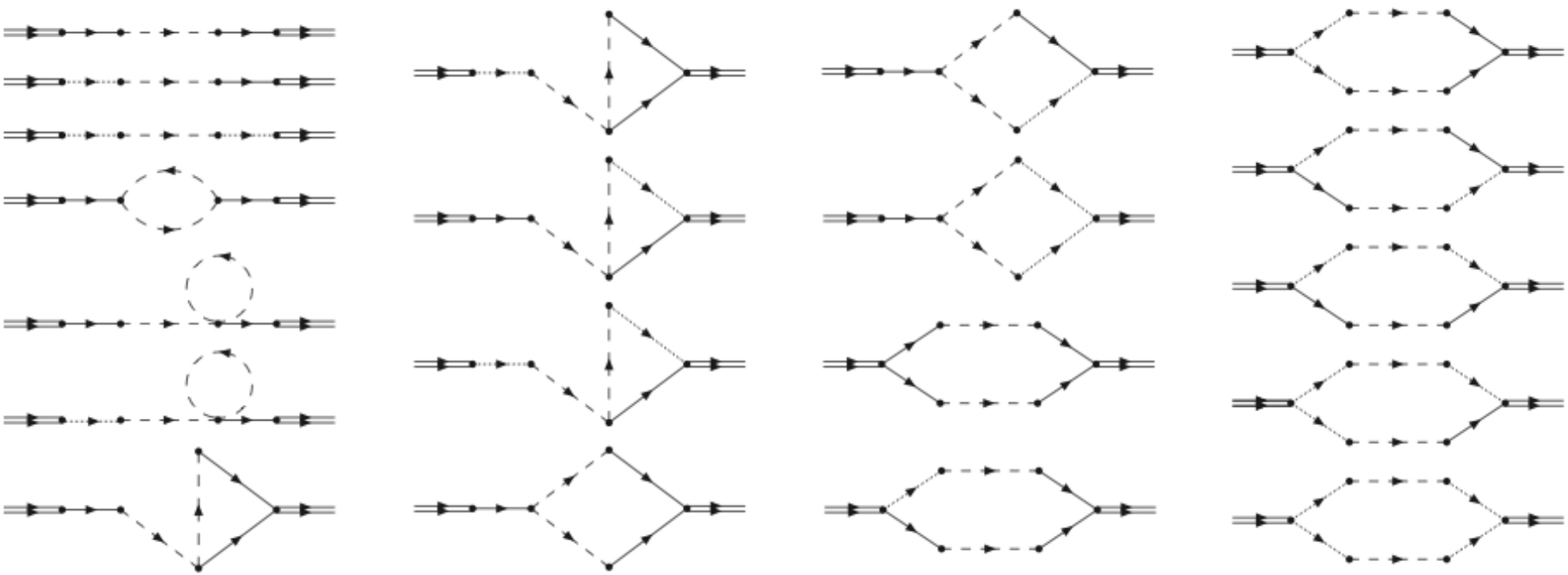
$$P(k) \;\; = \;\; \int d^3q \; e^{-i\mathbf{k}\cdot\mathbf{q}} \left[\int \frac{d\lambda_1}{2\pi} \frac{d\lambda_2}{2\pi} \, F(\lambda_1) F(\lambda_2) \left\langle e^{i[\lambda_1 \delta_L(\mathbf{q}_1) + \lambda_2 \delta_L(\mathbf{q}_2)] + i\mathbf{k}\cdot\Delta\mathbf{\Psi}} \right\rangle - 1 \right]$$

- which can be massaged with the same tricks as we used for the mass.
- If we assume halos/galaxies form at peaks of the initial density field ("peaks bias") then explicit expressions for the integrals of F exist.

The answer

$$P_{\rm obj}^{(s)} = e^{-\left[1+f(f+2)\mu^2\right]k^2\Sigma^2/2}\left[\left(b+f\mu^2\right)^2 P_L + \sum_{n,m}\mu^{2n}f^m E_{nm}\right]$$

Note angle dependence of damping.

Zel'dovich damping

Mode coupling terms up to E_{44}. These terms involve b_1 and b_2.