Gravity on Cosmic Scales

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1 Introduction

These notes summarize my lectures at *Cosmology on the Beach 2011*. For reference, please see arXiv/astro-ph:9503125 (LesHouches), 9506072 (MaB95), 0604485 (B06a) and 0607319 (B06b).

Starting from Newtonian gravity on terrestrial scales, we can derive the basics of the standard cosmological model in general relativity (GR). Consider a homogeneous medium with uniform expansion. Arbitrarily select a point as the origin about which the mass distribution is spherically symmetric and the motion is isotropic. The gravitating mass density is $\bar{\rho}(t)$. Label one of the particles in the expanding mass distribution by its initial radius R_0 at time t_0; its trajectory $R(t)$ obeys the energy conservation law

$$\frac{1}{2}\left(\frac{dR}{dt}\right)^2 - \frac{G}{R}\left(\frac{4\pi}{3}\bar{\rho}R^3\right) = E \equiv -\frac{1}{2}KR_0^2 \ . \tag{1}$$

Now require that the expansion be uniform, implying that two different test particles labeled 1 and 2 must move away from the origin in proportion to their initial radii. This implies that the radius of every particle obeys $R(t) = R_0 a(t)$ with a different R_0 for each particle but a universal $a(t)$ for all particles. Then equation (1) becomes

$$\left(\frac{da}{dt}\right)^2 - \frac{8\pi}{3}G\bar{\rho}a^2 = -K$$

where K and $a(t)$ are the same for all particles. Immediately we obtain the Friedmann equation

$$H^2 = \frac{8\pi}{3}G\bar{\rho} - \frac{K}{a^2} \ , \quad H \equiv \frac{d\ln a}{dt} \ . \tag{2}$$

Conventionally we normalize quantities so that $a = 1$ and $H = H_0$ today while $t = 0$ at the big bang ($a = 0$). Note that in an expanding universe $a(t)$ is monotonic and therefore one may use either t or a as a time variable. Cosmologists often use redshift z as the time variable because it is directly measurable (neglecting departures from uniform expansion); $1 + z = a^{-1}$.

The #1 goal of observational cosmology is to determine $\bar{\rho}(a)$ by measuring $H(a)$ in the Friedmann equation.

Frequently cosmologists assume that the composition of $\bar{\rho}(a)$ is a sum of w-fluids. A w-fluid is a medium whose pressure is proportional to its density: $p = w\rho$ (with units chosen so that the speed of light is $c = 1$). Assuming further that the different components separately conserve energy and that there is no exchange of heat between them, the expansion of each fluid is adiabatic,

$$d(\rho a^3) + p da^3 = 0 \ .$$

Inserting the equation of state $p = w\rho$ and integrating yields

$$\rho(a) \propto a^{-3(1+w)} \ .$$

The cases of interest are $w = 0$ for matter (for hydrogen gas at 10^6 K, $w \sim 10^{-7}$ can be neglected), $w = 1/3$ for radiation (photons and other ultrarelativistic particles), and $w = -1$ for a cosmological constant. We can now parameterize the mass density in terms of closure density fractions Ω_x and equations of state w_x:

$$\bar{\rho}(a) = \frac{3H_0^2}{8\pi G} \sum_x \Omega_x a^{-3(1+w_x)} \ . \tag{3}$$

For accelerated expansion, da/dt increases with time. Assuming the Friedmann equation (2), this requires $\bar{\rho}a^2$ to increase with a and therefore $\rho + 3P < 0$. Such matter is exotic. The standard cosmological model, ΛCDM, has (at low redshift) 3 non-interacting components: atoms with $w = 0$, cold dark matter with $w = 0$, and dark energy with $w = -1$. The last component is equivalent to a cosmological constant,

$$\Lambda = \frac{8\pi}{3} G \rho_\Lambda \ . \tag{4}$$

These results raise an important question: Is cosmic acceleration caused by Λ, by some other exotic fluid, or does the Friedmann equation break down on distance scales of Gpc?

The #3 goal of observational cosmology is to test general relativity on distance scales from kpc to Gpc.

Our task would be simple if this test could be done using the measured $H(a)$. (Actually, this function is not very well measured; improving its determination for $0.3 < a \leq 1$ is a major goal of present and future surveys using supernovae, baryon acoustic oscillations, and other distance indicators.) However, measuring $H(a)$ alone cannot distinguish GR from alternatives.

As an example, the class of theories known as $f(R)$ models replaces the Ricci scalar $R = g^{\mu\nu} R_{\mu\nu}$ in the Einstein-Hilbert action of gravity with $R + f(R)$. As a result, the Einstein field equations are modified to

$$(1 + f_R) R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}(R + f) + (g_{\mu\nu}\Box - \nabla_\mu \nabla_\nu) f_R = 8\pi G T_{\mu\nu} \tag{5}$$

where

$$f_R \equiv \frac{df}{dR} \quad \text{and} \quad \Box \equiv g^{\alpha\beta}\nabla_\alpha\nabla_\beta \ .$$

For a spatially flat ($K = 0$), homogeneous, uniformly expanding spacetime, the Friedmann equation is modified to

$$(1 + f_R)H^2 + \frac{1}{6}f - \frac{1}{a^3}\frac{d^2a}{dt^2}f_R + \frac{H}{a}\frac{df_R}{dt} = \frac{8\pi}{3}G\bar{\rho} \ . \tag{6}$$

The ΛCDM model corresponds to $f = -2\Lambda$ (with $f_R = 0$). Note that for any $H(a)$, with $\bar{\rho} = \rho_0 a^{-3}$ and no dark energy, one can find a $f(R)$ that solves equation (6) (Song, Hu and Sawicki, arXiv:astro-ph/0610532).

The #1 goal of observational cosmology is founded on the untested premise of the Friedmann equation. Observations of cosmic acceleration can be explained in very different ways. For example, either $f(R) = 0$, the Friedmann equation is valid, and there is dark energy (a fluid with $w < -\frac{1}{3}$), or $f(R) \neq 0$, the Friedmann equation is modified, and there is no dark energy, only matter with $w = 0$.

How can these alternatives (and the intermediate cases) be distinguished? For this, one must turn to cosmological perturbations.

2 Cosmological Perturbation Theory

The starting point for cosmological perturbation theory is the perturbed Robertson-Walker line element:

$$ds^2 = a^2(\tau)[-(1 + 2\Phi)d\tau^2 + (1 - 2\Psi)dl^2] \ , \tag{7}$$

where τ is called conformal time and is related to the proper time t used above by

$$t = \int a(\tau)\, d\tau \ . \tag{8}$$

For later use, we define the conformal expansion rate,

$$\mathcal{H} \equiv \frac{d\ln a}{d\tau} = aH \ . \tag{9}$$

The spatial line element dl^2 describes a constant-curvature 3-space in comoving coordinates. Using spherical polar coordinates (χ, θ, ϕ),

$$dl^2 = d\chi^2 + r^2(\chi)\left(d\theta^2 + \sin^2\theta d\phi^2\right) \ , \tag{10}$$

where the angular-diameter distance is given by

$$r(\chi) = \begin{cases} \chi, & K = 0 \\ \frac{1}{\chi\sqrt{K}}\sin(\chi\sqrt{K}), & K > 0 \\ \frac{1}{\chi\sqrt{-K}}\sinh(\chi\sqrt{-K}), & K < 0 \ . \end{cases} \tag{11}$$

Note that K has units of inverse length squared; it is not normalized to ± 1 or 0. Throughout this section, we allow K to take any value. In the familiar flat case, $K = 0$, Cartesian comoving coordinates (x_1, x_2, x_3) may be used, with $dl^2 = dx_1^2 + dx_2^2 + dx_3^2$. The name "Conformal Newtonian" is chosen because in the flat case equation (7) is conformal to (i.e., differs only by an overall multiplicative function from) the weak-field line element corresponding to the Newtonian limit of GR. The only new physics is the distinction made between the two potentials $\Phi(\tau, \boldsymbol{x})$ (the Newtonian potential) and $\Psi(\tau, \boldsymbol{x})$ (the curvature potential). They are assumed to be small; all terms quadratic in Φ, Ψ and their derivatives will be neglected in the following.

Equation (7) neglects vector (rotational) and tensor (gravitational wave) perturbations. This is justified in the weak-field limit where the scalar, vector, and tensor perturbations decouple. See LesHouches for details.

The conformal Newtonian gauge is one of infinitely many possible choices in general relativity. The metric perturbations play the role of gauge potentials similar to A_μ in electromagnetism. In general relativity, gauge invariance corresponds to coordinate invariance (also called general covariance). Infinitesimal coordinate transformations change tensor components. One must therefore be careful that physical quantities are gauge-invariant. Two approaches are possible. One is to define and work with explicitly gauge-invariant quantities (similar to the elecric and magnetic fields). This is often cumbersome because the results depend in general on the manner in which spacetime is sliced by coordinates (or equivalently, on the worldlines of preferred observers); there is no preferred slicing. Another approach, followed here, is to impose restrictions on the coordinates used ("gauge-fixing"). Besides the Conformal Newtonian gauge, another popular choice is synchronous gauge, in which the time part of the metric is unperturbed. Explicit transformations between these gauges are given by MaB95. Synchronous gauge is often a good choice for numerical integration but is inferior to Conformal Newtonian gauge for analytical work and for developing physical intuition.

2.1 Matter Perturbations

In addition to the line element, we require the sources of mass-energy-momentum that generate metric perturbations according to the Einstein field equations. The perturbed energy-momentum tensor components are

$$T^0{}_0 = -(\bar{\rho} + \delta\rho) \ , \tag{12a}$$

$$T^0{}_i = -(\bar{\rho} + \bar{p})\nabla_i u \ , \tag{12b}$$

$$T^i{}_j = \delta^i{}_j(\bar{p} + \delta p) + \frac{3}{2}(\bar{\rho} + \bar{p})\left(\nabla^i\nabla_j - \frac{1}{3}\delta^i{}_j\Delta\right)\pi \ . \tag{12c}$$

Here, ∇_i is the covariant derivative with respect to the spatial line element (10); raising its index with the inverse spatial metric gives ∇^i. The Laplacian in Euclidean space becomes the Laplace-Beltrami operator $\Delta = \nabla^i\nabla_i$ in curved space. Scalar perturbations

are characterized by four spatial scalar fields: the density perturbation $\delta\rho$, the velocity potential u, the pressure perturbation δp and the shear stress potential π. These all vanish for the background Robertson-Walker spacetime. In practice, it is convenient to replace $\delta\rho$ and δp by dimensionless number density and entropy density perturbations, defined as follows:

$$\delta = \frac{\delta\rho}{\bar{\rho}+\bar{p}} \ , \quad \sigma = \frac{\delta p - c_s^2 \delta\rho}{\bar{\rho}+\bar{p}} \ , \tag{13}$$

where the effective sound speed squared is defined by $c_s^2 \equiv d\bar{p}/d\bar{\rho}$.

The composition of the universe is often approximated as a multicomponent fluid (B06a). The net perturbations are weighted by each fluid's enthalpy fraction,

$$f_x(a) \equiv \frac{(\bar{\rho}+\bar{p})_x}{\sum_x (\bar{\rho}+\bar{p})_x} = \frac{(\bar{\rho}+\bar{p})_x}{\bar{\rho}+\bar{p}} \ . \tag{14}$$

Specifically,

$$\delta_{\rm tot} = \sum_x f_x \delta_x \ , \ \ u_{\rm tot} = \sum_x f_x u_x \ , \ \ \sigma_{\rm tot} = \sum_x f_x [\sigma_x + (c_x^2 - c_s^2)\delta_x] \ , \ \ \pi_{\rm tot} = \sum_x f_x \pi_x \ . \tag{15}$$

As we will see, the Einstein equations relate the metric perturbations Φ and Ψ to the background quantities $a(\tau)$ and $\bar{\rho}+\bar{p}$ and to the total fluid perturbations $\delta_{\rm tot}$, $u_{\rm tot}$, $\sigma_{\rm tot}$ and $\pi_{\rm tot}$.

Using energy conservation, one may show $c_s^2 = \sum_x f_x c_x^2$ which, when combined with (15), allows us to write

$$\sigma_{\rm tot} = \sum_x f_x [\sigma_x + c_x^2 (\delta_x - \delta_{\rm tot})] \ . \tag{16}$$

Thus there are two sources of entropy perturbations. Imperfect fluids can have intrinsic entropy perturbations σ_x. In addition, variations in composition (i.e., varying number density ratios with $\delta_x \neq \delta_{\rm tot}$) also correspond to entropy perturbations. For perfect w-fluids, $\sigma_x = 0$ and $c_x^2 = w_x$.

During much of the cosmic evolution, the composition can be approximated by the sum of two perfect fluids, one with $w = 0$ (cold dark matter and atoms, labeled m) and one with nonzero w (radiation or dark energy, labeled x). In this case,

$$f_m = \frac{\bar{\rho}_m}{\bar{\rho}+\bar{p}} \ , \quad f_x = 1 - f_m \ , \quad \sigma_{\rm tot} = f_x f_m c_s^2 (\delta_x - \delta_m) \ . \tag{17}$$

A cosmological constant has $f_x = 0$ and contributes nothing to perturbations. Dark energy with $w_x \neq -1$ has $f_x \propto (1 + w_x)$ leading to nonzero $\sigma_{\rm tot}$.

An example of an imperfect fluid is relativistic neutrinos after the weak interactions decouple at $T \sim 2$ MeV in the early universe. Prior to decoupling, the neutrino momentum distribution is isotropic in the local fluid rest frame. After decoupling, quadrupole anisotropies develop for neutrinos much as they do for photons after recombination.

Quadrupole anisotropy produces shear stress hence a nonzero π_ν. During the radiation-dominated era, initial curvature fluctuations induce fluctuations in the neutrino density and shear stress potential that are related by

$$\pi_\nu \approx \frac{2\delta_\nu}{45\mathcal{H}^2} \tag{18}$$

on scales larger than the Hubble length (MaBert95, with some change of notation). For smaller scales, higher-order anisotropies develop and it is necessary to integrate the collisionless Boltzmann equation for neutrinos (and, during and after recombination, the Boltzmann equation for photons).

These complications are handled by standard programs that calculate the evolution of cosmic perturbations (GRAFIC, CMBFAST, CAMB, etc).

Another case of some interest is a minimally coupled scalar field $\varphi = \varphi_0(\tau) + \varphi_1(\tau, \boldsymbol{x})$ with potential $V(\varphi)$. This case describes cosmic axions, simple inflationary models, and simple quintessence dark energy models. The energy density and pressure of the background scalar field are

$$\rho_\varphi = \frac{1}{2a^2}\dot{\varphi}_0^2 + V(\varphi_0) \ , \quad p_\varphi = \frac{1}{2a^2}\dot{\varphi}_0^2 - V(\varphi_0) \ , \tag{19}$$

where a dot is $\partial/\partial\tau$, yielding the equation of state

$$w_\varphi = \frac{\dot{\varphi}_0^2 - 2a^2 V(\varphi_0)}{\dot{\varphi}_0^2 + 2a^2 V(\varphi_0)} \tag{20}$$

and sound speed

$$c_\varphi^2 = 1 + \frac{2a^2}{3\eta\dot{\varphi}_0}\frac{dV}{d\varphi_0} \ . \tag{21}$$

The scalar field perturbations evolve exactly like a fluid with perturbations

$$\delta_\varphi = -\Phi + \frac{1}{\dot{\varphi}_0}\left[\dot{\varphi}_1 - \frac{3}{2}\mathcal{H}(1 - c_\varphi^2)\varphi_1\right] \ , \quad u_\varphi = \frac{\varphi_1}{\dot{\varphi}_0} \ , \quad \sigma_\varphi = (1 - c_\varphi^2)(\delta_\varphi + 3\mathcal{H}u_\varphi) \ , \quad \pi_\varphi = 0 \ . \tag{22}$$

A scalar field is not a perfect fluid — the equation of state in the Conformal Newtonian coordinate frame is not determined by the density alone but also depends on velocity perturbations. Although this would seem to violate Lorentz or Galilean invariance, it does not; the entropy perturbation σ is not evaluated in the fluid rest frame but rather in a coordinate frame through which energy is transported. It would be instructive to evaluate orthonormal components of the energy-momentum tensor in a frame in which the energy flux vanishes. Doing this calculation requires introducing the tetrad formalism, which goes beyond the level of these notes.

Speaking of evolution, energy-momentum conservation for each non-interacting fluid component, or for the total fluid, gives

$$\dot{\delta} + 3\mathcal{H}\sigma = 3\dot{\Psi} + \Delta u \ , \tag{23a}$$

$$\dot{u} + \mathcal{H}u = \Phi + c_s^2(\delta + 3\mathcal{H}u) + \sigma + (\Delta + 3K)\pi \ . \tag{23b}$$

(The variables δ, u, σ and π should have subscript tot or x as appropxiate; in the latter case c_s^2 is to be replaced by c_x^2.)

2.2 Einstein Equations and Their Solution

The above discussion is valid for any metric theory of gravity exhibiting general covariance with matter fields obeying local energy-momentum conservation — including GR, scalar-tensor theories, $f(R)$ theories and TeVeS. To complete the description of cosmological perturbation theory we need equations of motion for the metric perturbations Φ and Ψ.

In general relativity, substituting (7) and (12) into (5) with $f(R) = 0$ and linearizing yields

$$(\Delta + 3K)\Psi = \alpha(\delta_{\rm tot} + 3\mathcal{H}u_{\rm tot}) \ , \tag{24a}$$

$$\dot{\Psi} + \eta\Phi = \alpha u_{\rm tot} \ , \tag{24b}$$

$$\frac{1}{3}(\Psi - \Phi) = \alpha\pi_{\rm tot} \ , \tag{24c}$$

$$\ddot{\Psi} - K\Psi + \mathcal{H}(2\dot{\Psi} + \dot{\Phi}) + \left(2\dot{\mathcal{H}} + \mathcal{H}^2\right)\Phi + \frac{1}{3}\Delta(\Phi - \Psi) = \frac{\alpha\delta p}{\bar{\rho} + \bar{p}} \ , \tag{24d}$$

where

$$\alpha \equiv 4\pi G a^2(\bar{\rho} + \bar{p}) \ . \tag{25}$$

Equations (24)–(25) can be combined with (9) and (13) to give an alternative form of (24d) (B06a):

$$\frac{\alpha}{\mathcal{H}}\frac{\partial}{\partial\tau}\left[\frac{\mathcal{H}^2}{\alpha a^2}\frac{\partial}{\partial\tau}\left(\frac{a^2}{\mathcal{H}}\Psi\right)\right] - c_s^2\Delta\Psi = \alpha S \ , \quad S \equiv \sigma_{\rm tot} + \frac{3}{\mathcal{H}}\frac{\partial}{\partial\tau}\left(\mathcal{H}^2\pi_{\rm tot}\right) + \Delta\pi_{\rm tot} \ . \tag{26}$$

This equation is exact in general relativity with linear perturbation theory for open, curved, or flat backgrounds, and for arbitrary matter fields and dynamics.

Let's admire equation (26) for a moment. In contrast with equations (24a)–(24c), it has two time derivatives and therefore represents an equation of motion for the curvature potential $\Psi(\tau, \boldsymbol{x})$. The other potential Φ has no independent dynamics. On the other hand, the two potentials are constrained by equations (24a)–(24b) with the energy and momentum densities of the sources. These constraints are similar to the constraint on the longitudinal electric field that arises through charge conservation and electromagnetic gauge invariance.

Although the curvature potential Ψ obeys a modified wave equation, the wave speed c_s is the acoustic speed of adiabatic pressure perturbations and has nothing to do with the structure of spacetime itself. In the scalar mode of cosmological perturbations, all the dynamics occurs in the matter sector, and the metric responds to the matter. Only in the tensor mode does independent dynamics occur in the metric (gravitational radiation).

In Fourier space, $\Delta\Psi = -k^2\Psi$ where $k \geq 0$ is the comoving wavenumber. If $k^2 \ll k_{\rm J}^2$ where the comoving Jeans wavenumber is defined by

$$k_J \equiv \frac{c_s}{\mathcal{H}} \ , \tag{27}$$

the spatial derivative term is neglible and equation (26) can be integrated by quadratures:

$$\Psi(\tau, \mathbf{x}) = \kappa(\mathbf{x})\Psi_+(\tau) + C(\mathbf{x})\Psi_-(\tau) + \int_0^\tau [\Psi_+(\tau)\Psi_-(\tau') - \Psi_+(\tau)\Psi_-(\tau')]S(\tau', \mathbf{x})a^2(\tau')\, d\tau' \tag{28}$$

where

$$\Psi_-(\tau) = \frac{\mathcal{H}}{a^2} \ , \quad \Psi_+(\tau) = \frac{\mathcal{H}}{a^2}\int^\tau \frac{\alpha(\tau')a^2(\tau')}{\mathcal{H}^2(\tau')}\, d\tau' \ . \tag{29}$$

The long-wavelength solution is a combination of a "growing mode" proportional to an initial curvature perturbation $\kappa(\mathbf{x})$, a "decaying mode" proportional to $\Psi_-(\tau)$, and an "isocurvature" mode that vanishes for a perfect fluid (e.g., ΛCDM). The decaying mode is a gauge mode which can be eliminated by the coordinate transformation $\tau \to \tau + C(\mathbf{x})a^{-2}$.

The solution (28) is remarkably simple; moreover, it is accurate for length scales larger than the Jeans length, which in the case of CDM models makes it valid for all linear scales (e.g., wavelengths larger than about 20 Mpc).

We close by giving the solution for several other commonly used variables. First is the curvature perturbation κ of B06a:

$$\kappa = \frac{\mathcal{H}^2}{\alpha a^2}\frac{\partial}{\partial\tau}\left(\frac{a^2\Psi}{\mathcal{H}}\right) \ . \tag{30}$$

The equation of motion for κ is

$$\frac{\mathcal{H}^2}{\alpha a^2}\frac{\partial}{\partial\tau}\left(\frac{\alpha a^2}{\mathcal{H}^2}\frac{\partial\kappa}{\partial\tau}\right) - c_s^2\Delta\kappa = \frac{\mathcal{H}^2}{\alpha a^2}\frac{\partial}{\partial\tau}\left(\frac{\alpha a^2}{\mathcal{H}}S\right) \ . \tag{31}$$

This is exactly in the form of a damped, driven wave equation with damping rate

$$\Gamma_\kappa = \frac{d}{d\tau}\ln\left(\frac{\alpha a^2}{\mathcal{H}^2}\right) \ . \tag{32}$$

In a flat background this becomes

$$\Gamma_\kappa = \left[2 + 3(w - c_s^2)\right]\mathcal{H} \ .$$

The curvature perturbation κ is especially simple because for wavelengths longer than the Jeans length, $\kappa =$ constantin the absence of a source term.

A commonly used curvature variable was introduced by Lyth (PRD 31, 1792,1985; see also the textbook *Cosmological Inflation and Large-Scale Structure* by Liddle and Lyth):

$$\mathcal{R} = \Psi + \mathcal{H} u_{\rm tot} \ . \tag{33}$$

In a flat background, $\mathcal{R} = \kappa$ but in a curved background they differ. In both cases, for a flat universe dominated by a single w-fluid,

$$\kappa = \mathcal{R} = \frac{5+3w}{3+3w}\Psi \ \ \text{if} \ \ |K| \ll k^2 \ll \mathcal{H}^2 \ \ \text{and} \ \ \dot{w} = 0 \ . \tag{34}$$

When the equation of state changes, as happens, for example, during the radiation to matter transition, κ remains constant but the amplitude of Ψ changes for all wavelengths longer than the Jeans length by a constant factor (9/10 in the case of the radiation to matter transition).

2.3 Evolution of Matter Perturbations

Finally, we come to a more directly observable quantity than the metric potentials: the density perturbation itself, which is revealed by galaxy clustering. The natural density perturbation to use is not the coordinate density perturbation δ; on Hubble length scales it is dramatically affected by the choice of time-slicing. A better variable is a gauge-invariant density perturbation tied to the matter itself. Under an infinitesimal coordinate transformation $\tau \to \tau + u$, the velocity perturbation vanishes and the energy density perturbation transforms to $\delta\rho + 3(\bar{\rho}+\bar{p})\mathcal{H}u$. This quantity is therefore the energy density perturbation in the fluid rest frame and it can be used to define a gauge-invariant measureable density perturbation (Bardeen PRD 22, 1882, 1980; S06b):

$$\delta\rho_{\rm rest} = (\bar{\rho}+\bar{p})(\delta + 3\mathcal{H}u) \equiv \bar{\rho}\epsilon_m \equiv (\bar{\rho}+\bar{p})\nu \ . \tag{35}$$

Combining equations (24a), (25) and (26), we obtain the following exact equation in linear perturbation theory of GR

$$\ddot{\nu} + (1-3c_s^2)\mathcal{H}\dot{\nu} - \left[(1+3c_s^2)\alpha + \frac{3}{\mathcal{H}}\frac{d}{d\tau}\left(c_s^2\mathcal{H}^2\right)\right]\nu - c_s^2\Delta\nu = (\Delta + 3K)S \ . \tag{36}$$

For the ΛCDM model with $c_s^2 = S = 0$, this reduces to the conventional Newtonian structure formation equation

$$\ddot{\nu} + \mathcal{H}\dot{\nu} = \alpha\nu \ \ \text{or equivalently} \ \ \ddot{\nu}_m + \mathcal{H}\dot{\nu}_m = 4\pi G a^2 \bar{\rho}_m \nu_m \ . \tag{37}$$

The solution to this equation has been approximated through its growth rate by Linder (astro-ph/0507263) as

$$\frac{d\ln\nu_m}{d\ln a} = \Omega(a)^\gamma \tag{38}$$

where $\Omega(a) \equiv 8\pi G\bar{\rho}_m(a)/3H^2(a)$ and γ is a constant growth index. Clearly the evolution of density perturbations contains information about the background cosmology and the law of gravity; γ is a good choice of variable to measure from the redshift evolution of galaxy clustering. Returning to the questions posted in the Introduction, we ask, what does it tell us?

Like the Friedmann equation, equation (37) contains information about the matter density: measurement of the Hubble expansion history combined with the growth rate $d\ln\nu_m/d\ln a$ enables determination of $\bar{\rho}_m(a)$. This provides an important consistency check with the Friedmann equation itself. However, does not provide a full test of GR. Any deviations from (37) combined with the Friedmann equation could be ascribed to exotic dark matter or dark energy with $S \neq 0$ or $c_s^2 \neq 0$. Moreover, as we will see later, modified gravity adds further degrees of freedom that will modify the evolution of dark matter perturbations. Distinguishing modifications of gravity from exotic dark matter or dark energy requires additional tests.