Gravity on Cosmic Scales

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References

EBNotes.pdf at the conference website

arXiv/astro-ph:

9503125, 9506072, 0604485 and 0607319.

Overview

1: Cosmological Perturbation Theory in GR

2: Cosmological Perturbation Theory in other Metric Theories

3: Cosmological Scale Tests of Gravity

Elementary cosmology

Cosmic scale factor a(t)

In Newtonian gravity and GR,

$$H^2 = \frac{8\pi}{3} G\bar{\rho} - \frac{K}{a^2}\ , \quad H \equiv \frac{d \ln a}{dt}\ .$$

where K is spatial curvature (units 1/length^2).

The #1 goal of observational cosmology is to determine ρ(a) by measuring H(a) in the Friedmann equation.

w-fluids

If equation of state p=wρ with w=constant and c=1, energy conservation implies

$$\rho(a) \propto a^{-3(1+w)}$$

Superposition of w-fluids:

$$\bar{\rho}(a) = \frac{3H_0^2}{8\pi G} \sum_x \Omega_x a^{-3(1+w_x)} \quad .$$

In particular, for w=−1,

$$\Lambda = \frac{8\pi}{3} G \rho_\Lambda$$

What causes da/dt to increase?

Cosmological constant

Some other exotic field with $\rho + 3p < 0$

Breakdown of the Friedmann equation: GR needs modification on cosmological scales

The #3 goal of observational cosmology is to test general relativity on distance scales from kpc to Gpc.

Example: f(R) gravity

Modified Einstein equations

$$(1+f_R)R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}(R+f)+(g_{\mu\nu}\Box-\nabla_\mu\nabla_\nu)f_R = 8\pi G T_{\mu\nu}$$

where

$$f = -2\Lambda + c_2 R^2 + \cdots \ , \quad f_R \equiv \frac{df}{dR}$$

Modified Friedmann equation (if K=0)

$$(1+f_R)H^2 + \frac{1}{6}f - \frac{1}{a^3}\frac{d^2a}{dt^2}f_R + \frac{H}{a}\frac{df_R}{dt} = \frac{8\pi}{3}G\bar{\rho}$$

Dark Energy or Modified Gravity?

Any expansion history can be accounted for by

1. Dark energy with $w < -1/3$ and GR
2. No dark energy; instead modify GR

(Theorem:
Song, Hu & Sawicki, astro-ph/0610532)

How to test gravity?
Perturbations

$$ds^2 = a^2(\tau)[-(1+2\Phi)d\tau^2 + (1-2\Psi)dl^2]$$

$$t = \int a(\tau)\, d\tau$$

$$dl^2 = d\chi^2 + r^2(\chi)\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

Conformal Newtonian Gauge

for scalar perturbations

(Common alternative: synchronous gauge

Numerically fine, analytically inferior)

Source of gravity

$$
\begin{aligned}
T^0{}_0 &= -(\bar{\rho} + \delta\rho) \; , \\
T^0{}_i &= -(\bar{\rho} + \bar{p})\nabla_i u \; , \\
T^i{}_j &= \delta^i{}_j(\bar{p} + \delta p) + \frac{3}{2}(\bar{\rho} + \bar{p})\left(\nabla^i\nabla_j - \frac{1}{3}\delta^i{}_j\Delta\right)\pi \; .
\end{aligned}
$$

Introduce dimensionless number density and entropy density perturbations:

$$
\delta = \frac{\delta\rho}{\bar{\rho} + \bar{p}} \; , \quad \sigma = \frac{\delta p - c_s^2\delta\rho}{\bar{\rho} + \bar{p}} \; ,
$$

Note $u = \theta/k^2$

Fluid equations

Continuity:

$$\dot{\delta} + 3\mathcal{H}\sigma = 3\dot{\Psi} + \Delta u$$

Euler/Bernoulli:

$$\dot{u} + \mathcal{H}u = \Phi + c_s^2(\delta + 3\mathcal{H}u) + \sigma + (\Delta + 3K)\pi$$

Perturbed Einstein equations

$$
\begin{aligned}
(\Delta + 3K)\Psi &= \alpha(\delta_{\text{tot}} + 3\mathcal{H}u_{\text{tot}}) \ , \\
\dot{\Psi} + \eta\Phi &= \alpha u_{\text{tot}} \ , \\
\frac{1}{3}(\Psi - \Phi) &= \alpha\pi_{\text{tot}} \ ,
\end{aligned}
$$

$$
\ddot{\Psi} - K\Psi + \mathcal{H}(2\dot{\Psi} + \dot{\Phi}) + \left(2\dot{\mathcal{H}} + \mathcal{H}^2\right)\Phi + \frac{1}{3}\Delta(\Phi - \Psi) = \frac{\alpha\delta p}{\bar{\rho} + \bar{p}} \ ,
$$

where

$$
\alpha \equiv 4\pi G a^2(\bar{\rho} + \bar{p}) \ .
$$

Beautiful combination

$$\frac{\alpha}{\mathcal{H}}\frac{\partial}{\partial\tau}\left[\frac{\mathcal{H}^2}{\alpha a^2}\frac{\partial}{\partial\tau}\left(\frac{a^2}{\mathcal{H}}\Psi\right)\right] - c_s^2\Delta\Psi = \alpha S \ ,$$

$$S \equiv \sigma_{\mathrm{tot}} + \frac{3}{\mathcal{H}}\frac{\partial}{\partial\tau}\left(\mathcal{H}^2\pi_{\mathrm{tot}}\right) + \Delta\pi_{\mathrm{tot}} \ .$$

**This is exact in linear PT of GR for any background and any matter fields.
Note that the source vanishes for CDM or a perfect fluid!**

Jeans instability

Jeans wavenumber $k_J = c_s/H$

For CDM models, this corresponds to

$\sim(10\ \text{Mpc})^{-1}$

For $k \ll k_J$, the gravitational potential has an exact quadrature solution

Curvature potential

$$\Psi(\tau, \mathbf{x}) = \kappa(\mathbf{x})\Psi_+(\tau) + C(\mathbf{x})\Psi_-(\tau) + \Psi_p(\tau, \mathbf{x})$$

$$\Psi_-(\tau) = \frac{\mathcal{H}}{a^2}$$

$$\Psi_+(\tau) = \frac{\mathcal{H}}{a^2}\int^\tau \frac{\alpha(\tau')a^2(\tau')}{\mathcal{H}^2(\tau')}\,d\tau'$$

$$\Psi_p = \int_0^\tau [\Psi_+(\tau)\Psi_-(\tau') - \Psi_+(\tau)\Psi_-(\tau')]S(\tau', \mathbf{x})a^2(\tau')\,d\tau'$$

Other curvature variables

$$\kappa = \frac{\mathcal{H}^2}{\alpha a^2}\frac{\partial}{\partial\tau}\left(\frac{a^2\Psi}{\mathcal{H}}\right)$$

$$\mathcal{R} = \Psi + \mathcal{H}u_{\mathrm{tot}}$$

For $k << k_J$, $\kappa=\mathcal{R}$ is constant

For a w-fluid, $\kappa = \mathcal{R} = \frac{5+3w}{3+3w}\Psi$

Density perturbation evolution

Conventional equation (Newtonian):

$$\ddot{\delta}_c + \mathcal{H}\dot{\delta}_c = 4\pi G a^2 \bar{\rho}\delta_c$$

But this is inconsistent with previous results!

$$\ddot{\delta}_c + \mathcal{H}\dot{\delta}_c = 4\pi G a^2 \bar{\rho}(\delta_{\text{tot}} + 3\mathcal{H}u_{\text{tot}}) + 3(\ddot{\Psi} + \mathcal{H}\dot{\Psi})$$

Gauge? Frame?

Gauge-invariant density perturbation

Bardeen 1980 $\varepsilon_m/(1+w) = \nu = \delta + 3\mathcal{H}u$

This is the number density perturbation in the fluid rest frame (the “N-body” gauge?)

$$(\Delta + 3K)\Psi = 4\pi G a^2 \bar{\rho}\nu_{\rm tot}$$

In pure CDM models, the simple matter evolution equation is valid.

In ΛCDM and others, it may not be!