• Gravity
general relativity, but
Newtonian approximation
in expanding space usually
sufficient

• dark matter is collisionless
  (described by Vlasov equation)

• Monte-Carlo integration as
  an N-body system

• 3N coupled, non-linear differential
  equations of second order

• Hydrodynamics
  shock waves
  radiation processes
  star formation
  supernovae
  black holes, etc...

• Problems:
  N is very large
  All equations are coupled
to each other
Two conflicting requirements complicate the study of hierarchical structure formation:

- **Want small particle mass** to resolve internal structure of halos.
- **Want large volume** to obtain representative sample of universe.

**Problems due to a small box size:**
- Fundamental mode goes non-linear soon after the first halos form. Simulation cannot be meaningfully continued beyond this point.
- No rare objects (the first halo, rich galaxy clusters, etc.).
- At any given time, halos exist on a large range of mass-scales!

**Problems due to a large particle mass:**
- Physics cannot be resolved.
- Small galaxies are missed.

**DYNAMIC RANGE IN COSMOLOGICAL SIMULATIONS**

- Need large $N$, where $N$ is the particle number.
• Computers double their speed every 18 months

• A naive N-body force calculation needs $N^2$ op's

• Simulations double their size every 16.5 months

• $N = 10^{10}$ should have been reached in 2008

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**Moore's Law for Cosmological N-body Simulations**

- *direct summation*
- $P^3M$ or $AP^3M$
- distributed-memory parallel Tree
- parallel or vectorized $P^3M$
- distributed-memory parallel TreePM

---

• The Universe has structure on all scales...

• need large $L$

• and spatial adaptivity
Idea: Group distant particles together, and use their multipole expansion.

The $N^2$-scaling of direct summation puts serious limitations on $N$...

But we want $N \sim 10^6 - 10^{10}$ for collisionless dynamics of dark matter!

Only $\sim \log(N)$ force terms per particle.
**The Tree-PM method (Gadget-2)**

- **Idea:** Compute the long-range force with the PM algorithm, and only a local short-range force with the tree.

- Let's split the potential in Fourier space into a long-range and a short-range part:

\[
\phi_{\text{long}}^{\text{long}} = \phi_k \exp(-k^2 r_s^2)
\]

\[
\phi_{\text{short}}^{\text{short}} = \phi_k \left[1 - \exp(-k^2 r_s^2)\right]
\]

- **Solve with PM-method**
  - CIC mass assignment
  - FFT
  - Multiply with kernel
  - FFT backwards
  - Compute force with 4-point finite difference operator
  - Interpolate forces to particle positions

- **Solve in real space with TREE**
  - Tree has to be walked only locally
  - \( \sim 5 r_s \)

- **Volker Springel**

- **Advantages of this algorithm include:**
  - Accurate and fast long-range force
  - No force anisotropy
  - Speed is insensitive to clustering (as for tree algorithm)
  - No Ewald correction necessary for periodic boundary conditions
### Cosmological hydrodynamics

- The baryons in the universe can be modelled as an **ideal gas**

#### Basic Equations

- **Euler equation:**
  \[
  \frac{dv}{dt} = -\frac{\nabla P}{\rho} - \nabla \Phi
  \]

- **Continuity equation:**
  \[
  \frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} = 0
  \]

- **First law of thermodynamics:**
  \[
  \frac{du}{dt} = -\frac{P}{\rho} \nabla \cdot \mathbf{v} - \frac{\Lambda(u, \rho)}{\rho}
  \]

- **Equation of state of ideal monoatomic gas:**
  \[
  P = (\gamma - 1) \rho u, \quad \gamma = \frac{5}{3}
  \]
Differing methods to discretize a fluid:

- **Eulerian**
  - Discretize space
  - Representation on a mesh (volume elements)
  - Principal advantage: high accuracy (shock capturing), low numerical viscosity

- **Lagrangian**
  - Discretize mass
  - Representation by fluid elements (particles)
  - Principal advantage: resolution adjusts automatically to the flow

**Cosmological hydrodynamics**
The origin of cosmic structure

- Inflation
  - Quantum fluctuations:
    \[ |\delta_k|^2 \propto k^n \]
    \[ n \approx 1 \]
  - Gaussian amplitudes
  - Flat universe
  -\[ P(k) = A k^n \]
  - Damping (nature of dark matter)

- Damping:
  - Meszaros damping
  - Free streaming

- Large scales
  - superclusters, clusters, galaxies
  -\[ R_h(t_{eq}) \]
  - Cold dark matter (e.g., neutralino)

- Small scales
  -\[ \delta \rho / \rho \]
  -\[ k^2 \]

- Hot DM (e.g., ~30 ev neutrino)
  - Top-down formation

- Cold DM (e.g., ~neutralino)
  - Bottom-up (hierarchical)
• Neutrino (hot) dark matter

• Free-streaming length so large that superclusters form first and galaxies are too young

• $\Omega_\nu = 1$ ($m_\nu = 30$ ev)

• Frenk, White & Davis '83
The CFA redshift survey

1982: 2543 galaxies, magnitude limited to $m_z = 14.5$

Davis, Huchra, Latham & Tonry ‘82
- Neutrino (hot) dark matter

- $\Omega_\nu = 1$ ($m_\nu = 30$ ev)

- Free-streaming length so large that superclusters form first and galaxies are too young

- Neutrinos cannot make an appreciable contribution to $\Omega$ and $m_\nu \ll 10$ ev

- CfA redshift survey

- Frenk, White & Davis '83
The origin of cosmic structure

- **Inflation**
- **Meszaros damping**
- **Free streaming**
- **Large scales**
- **Small scales**
- **P(\kappa)\approx1**
- **\delta\rho/\rho^2**
- **Superclusters, clusters, galaxies**
- **\lambda_Cut\propto m_\nu^{-1}**
- **\mathbb{Q}\Upsilon(k,\Omega\nu^2)**
- **Top-down formation**
- **Cold DM (eg ~neutralino)**
- **Bottom-up (hierarchical)**
- **Hot DM (eg ~30 ev neutrino)**
- **FLAT UNIVERSE**
- **P(k)=Ak^n**
- **Damping (nature of dark matter)**
- **Transfer function**
- **\mathbb{Q}\Upsilon(k,\Omega\nu^2)\propto m_\nu^{-1}**
- **Inflation (t~10^{-35} s)**
- **P(k)=A k^{n} T^2(k,t)**
- **Quantum fluctuations:**
  \[ |\delta_k|^2 \propto k^n \]
  - Gaussian amplitudes
  - \( n \approx 1 \)
• Non-baryonic dark matter cosmologies

• Neutrino dark matter produces unrealistic clustering

• Early CDM N-body simulations gave promising results

• In CDM structure forms hierarchically

• CfA redshift survey

• Davis, Efstathiou, Frenk & White '85

• CDM $\Omega = 0.2$

• Neutrinos $\Omega = 1$

• Davis, Efstathiou, Frenk & White '85
Galaxies trace mass
Dark matter

\[ \Omega = 1 \text{ CDM} \]

pec vels too large
wrong clustering

Fig. 16.—The projected distribution of all particles (left) and of the "galaxies" (right) the 2.5 \(\sigma\) peaks of the linear density distribution.
Biased galaxy formation

... or how to rescue $\Omega = 1$!  

DEFW '85

Dark matter

Galaxies

- High peaks of density field
  - Are more strongly clustered than the mass
Bias is inevitable for rare systems

- Peak-background split: \( \delta_c \rightarrow \delta_c - \varepsilon \);
- \( n(m) \rightarrow n(m) + \left( \frac{dn}{d\nu} \right) \left( \frac{d\nu}{d\varepsilon} \right) \varepsilon = n(m) \left[ 1 + b\varepsilon \right] \)
- Bias: \( \xi \rightarrow b^2\xi \) depends on halo mass

\[ b(M) = 1 + \left( \nu^2 - 1 \right) / \delta_c \]
\[ \nu \equiv \delta_c / \sigma(M) \]
SCDM compared to CfA-2 z-survey

White, Frenk, Davis, Efstathiou ‘87
Angular 2-pt correlation function

\( \Omega = 1 \) CDM under strain

Maddox, Efstathiou, Sutherland & Loveday ‘90
The end of standard ($\Omega_{\text{matter}} = 1$) CDM ... or why $\Omega_{\text{matter}}$ cannot be 1
X-ray emission from hot plasma in clusters

- X-rays $\Rightarrow$ gas mass
- Photometry $\Rightarrow$ stellar mass
- Gas in hydrostatic equilibrium so X-rays $\Rightarrow$ (or lensing) $\Rightarrow$ total gravitating mass

About 90% of baryons in clusters are in hot gas

Perseus $\ (z=0.0183)\
Hydra A $\ (z=0.054)\
A2052 $\ (z=0.0348)\$

Images from David Buote

Institute for Computational Cosmology
\[ \Omega = \frac{\text{mass in baryons}}{\text{total mass}} \]

\[ f_b = \gamma \frac{\Omega_b}{\Omega} \]

- \( f_b \) is the baryon fraction in clusters.
- \( \Omega_b \) is the baryon density parameter.
- \( \Omega \) is the total density parameter.
- \( \gamma \) is a correction factor.
- \( \gamma = 1 \) if the baryon fraction has the universal value.
- \( \gamma = 0.9 \) from simulations.

- In clusters, matter that has fallen in is still in the cluster (\( r_{\text{vir}} \sim r_{\text{non-linear}} \)).
- Baryon fraction in clusters \( \approx \) baryon fraction of the universe.

- \( M_b = M_{\text{gas}} + M_{\text{stars}} \)
- White, Navarro, Evrard & Frenk '93
\[ f_b = \frac{M_b}{M_{tot}} = \gamma \frac{\Omega_b}{\Omega_m} \]

where \( \gamma = 1 \) if \( f_b \) has the universal value

- simulations \( \Rightarrow \gamma = 0.9 \pm 10\% \)
- X-rays+lensing \( \Rightarrow f_b = (0.060h^{-3/2} +0.009) \pm 10\% \)
- BBNS, CMB \( \Rightarrow \Omega_b h^2 = 0.019 \pm 20\% \)
- HST \( \Rightarrow h = 0.7 \pm 10\% \)

\[ \Omega_m = \frac{\Omega_b \gamma}{f_b} = 0.31 \pm 0.12 \]

\( \Omega \) from the baryon fraction in clusters

baryon fraction in clusters \( \approx \) baryon fraction of universe

- White, Navarro, Evrard & Frenk
  Nature 1993

- Allen et al '04

- Baryon fraction in clusters \( \approx \) baryon fraction of universe
• $\Omega < 1$: open or flat universe?

\[ \Omega_{\text{matter}} < 1 \implies \begin{cases} 
\text{BBNS is wrong or} \\
\text{Universe is open or} \\
\text{Need cosmological constant if } \Omega_{\text{tot}} = 1 
\end{cases} \\
\text{White et al 1993} \]
The Virgo consortium

\( \Omega_m = 1 \)

\( \Omega_m = 0.2 \)

Jenkins et al. 1998

The VIRGO Collaboration 1996
(Some) evidence for dark energy
• Evidence for $\Lambda$ from high-z supernovae

• Distant SN are fainter than expected if expansion were decelerating

Riess et al '98
The Virgo consortium

Jenkins et al 1998

The VIRGO Collaboration 1996
- dalla Vechia, Jenkins & Frenk
- Comovin g coordinates
- $t = 0.06$ Gyr
• Testing the CDM model
The cosmic microwave background radiation (CMB) provides a window to the universe at $t \approx 3 \times 10^5$ yrs.

In 1992 COBE discovered temperature fluctuations ($\Delta T/T \approx 10^{-5}$) consistent with inflation predictions.
The CMB

199

200
• WMAP temp anisotropies in CMB

- The amplitude of the CMB ripples is exactly as predicted by inflationary cold dark matter theory
- The position of the first peak
- Hinshaw et al. 2006

Institute for Computational Cosmology
• The galaxy distribution evolves from fluctuations seen in CMB by gravitational amplification.
CMB:
- Convert angular separation to distance (and k) assuming flat geometry

- Extrapolate to z=0 using linear theory
The Virgo consortium

Jenkins et al 1998

The VIRGO Collaboration 1996
• Testing the $\Lambda$CDM paradigm with galaxy surveys

• Three tests:

• Origin of fluctuations, nature of DM, cosm.parms (Power spectrum of galaxy distribution)

• Gravitational instability (z-space distortions)

• Hierarchical galaxy formation (non-linear clustering)
Testing the $\Lambda$CDM paradigm with galaxy surveys

Three tests:

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- Gravitational instability (z-space distortions)
- Hierarchical galaxy formation (non-linear clustering)
The Millennium simulation

- Cosmological N-body simulation
  - 10 billion particles
  - 500 $h^{-1}$ Mpc box
  - $m_p = 8 \times 10^8$ $h^{-1}$ $M_\odot$
  - $\Omega = 1$; $\Omega_m = 0.25$; $\Omega_b = 0.045$;
    $h = 0.73$; $n = 1$; $\sigma_8 = 0.9$

- 20 $\times 10^6$ gals brighter than LMC

Carried out at Garching using L-Gadget by V. Springel

- 27 Tbytes of data

- UK, Germany, Canada, US collaboration

Simulation data available at:
- http://www.mpa-garching.mpg.de/Virgo
- www.durham.ac.uk/virgo

Pictures and movies available at:
- Nature, June/05
Regatta supercomputer of the RZG

January 2005

- 10 billion particles

2005
Regatta supercomputer of the RZG

2005

1983

• The Millennium simulation

Institute for Computational Cosmology
Hubble-Volume Simulation
1.000.000.000 particles
The non-linear mass power spectrum is accurately determined by the Millennium simulation over large range of scales.

The mass power spectrum at:
- $z=0$
- $z=1$
- $z=3.05$
- $z=7$
- $z=14.9$

- Linear growth
- Non-linear evolution

$\Delta^2(k)$ vs. $k [h/\text{Mpc}]$
• Dark matter concentrations amplify images of background galaxies and distort them into arc-like features

Abell 2218  \( z = 0.17 \)
<\gamma^2> is the mean square gravitational shear of background galaxy images within circles of radius \( \theta \).

- It is proportional to the mean square lensing mass within these circles.

- On scales of a few arcmin the signal is dominated by \textit{nonlinear} DM clustering, i.e. by the dark halos of galaxies and galaxy groups.

Van Waerbeke et al 01
2003: The 2dF Galaxy Redshift Survey

221,000 redshifts
- **Cosmological model**
  - \((\Omega_m, \Omega_\Lambda, h)\); dark matter

- **Primordial fluctuations**
  - \(\delta\rho/\rho(M, t)\)

  - **Dark matter halos**
    - (N-body simulations)

- **Gas processes**
  - (cooling, star formation, feedback)

  - **Gasdynamic simulations**
  - **Semi-analytics**

- **Formation and evolution of galaxies**

  - **Well established**
  - **Well understood**
The abundance of dark halos

ΛCDM Hubble Volume Simulation

Millennium run

3000 Mpc/h

Virgo consortium

Jenkins et al '01

Halo mass function

ΛCDM
The Density Profile of Cold Dark Matter Halos

Halo density profiles are independent of halo mass & cosmological parameters.

There is no obvious density plateau or `core' near the centre.

(Navarro, Frenk & White ‘97)

More massive halos and halos that form earlier have higher densities (bigger $\delta$)
The halo mass function and the galaxy luminosity function have different shapes.

Complicated variation of $M/L$ with halo mass.

Benson, Bower, Frenk, Lacey, Baugh & Cole '03
Modelling galaxy formation

- **Aim:** follow history of galaxy formation *ab initio*, i.e starting from a cosmological model for structure formation so as to predict observables

- **Main Physical processes:**
  - Assembly of dark matter halos
  - Shock-heating and radiative cooling of gas within halos
  - Star formation and feedback
  - Production & mixing of metals
  - Evolution of stellar populations
  - Dust extinction & emission
  - Black hole formation, AGN feedback
  - Galaxy mergers
Shock-induced burst

stars

gas
$z = 0$ Dark Matter

Springel et al. 05
Modelling galaxy formation

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  - Galaxy mergers

Semi-analytical model
  - Parametrize physics
  - Set of coupled differential equations
  - Predict galaxy prop. (lum, mass, B/T, radius, metallicity etc) and evolution with redshift